Tensor Operators are sets of operators that transform among each other in a special way under rotations.

Recall that vectors in rectangular coordinates obey:

\[ V_i \rightarrow \sum_{j=r,y,z} R_{ij} V_j \]

under a rotation parameterized by the orthogonal matrix \( R \).

States transform like:

\[ |\Psi\rangle \rightarrow D(R) |\Psi\rangle \]

where \( D(R) \) is the operator for rotation \( R \).

So, the expectation value of a vector operator transforms like:

\[
\langle \Psi | V_i | \Psi \rangle \rightarrow \langle \Psi | D^+(R) V_i D(R) | \Psi \rangle \\
= \langle \Psi | \sum_j R_{ij} V_j | \Psi \rangle 
\]

(for any state \( |\Psi\rangle \))

So, \( D^+(R) V_i D(R) = \sum_j R_{ij} V_j \)

For an infinitesimal rotation \( \epsilon \) about an axis direction \( \hat{n} \),

\[ D(R) = 1 - i \epsilon \hat{J} \cdot \hat{n} \]

Plug this into previous equation:

\[ V_i = \frac{i \epsilon}{\hbar} [V_i, \hat{J} \cdot \hat{n}] = \sum_j R_{ij} V_j \]

for \( \epsilon \) about \( \hat{n} \).

For example, if \( \hat{n} = \hat{z} \), then \( R = \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

So, look at \( i = x \):

\[ V_x = \frac{i \epsilon}{\hbar} [V_x, \hat{J}_z] = V_x - \epsilon V_y \]

Therefore, \( [V_x, \hat{J}_z] = -i \hbar V_y \).

Looking at other combinations the same way gives

\[
[V_i, J_j] = i \hbar \epsilon_{ijk} V_k
\]

(summed over \( k \))

Could take this as the defining requirement for \( V_i \) to be a vector.
(Can't just take any triplet of operators and call it a vector!)
For example, $V_i = J_i$ works. So does $V_i = R_i$, and $V_i = P_i$.

More generally, one can define Cartesian tensors as multi-index objects that satisfy:

\[ T_{ijk...} = R_{ii'}, R_{jj'}, R_{kk'}... T_{ij'j'k'...} \text{ (primed indices summed)} \]

The number of indices = rank of the tensor.

However, it is more convenient to discuss tensors in the spherical representation.

Recall how rotation operators act on angular momentum eigenstates:

\[ \mathcal{J}(R) |l, m\rangle = \sum_{m'} |l, m'\rangle \mathcal{O}^{(l)}_{m'm} (R) \]

This motivates the following...

**Definition** A spherical tensor operator of rank $k$ is a set of $2k + 1$ operators $T_{q}^{(k)}$ ($q = -k, -k + 1, ..., k - 1, k$) that satisfy:

\[ \mathcal{J}(R) T_{q}^{(k)} \mathcal{J}^+(R) = \sum_{q'}^{k} \mathcal{J}^{(k)}_{qq'}(R) T_{q'}^{(k)} \]

An alternate version is obtained by taking $\mathcal{J}(R) = 1 + \frac{i}{\hbar} \hat{\mathbf{J}} \cdot \hat{\mathbf{n}}$ again. Plugging in, we get

\[ [\hat{\mathbf{J}} \cdot \hat{\mathbf{n}}, T_{q}^{(k)}] = \sum_{q'} T_{q'}^{(k)} \langle k q' | \hat{\mathbf{J}} \cdot \hat{\mathbf{n}} | k q \rangle \]

Or, using $\hat{\mathbf{n}} = \hat{x}$ and then $\hat{\mathbf{n}} = \hat{x} \pm i \hat{y}$,

\[ [J_z, T_{q}^{(k)}] = k q \ T_{q}^{(k)} \]

\[ [J_\pm, T_{q}^{(k)}] = k \sqrt{(k + q)(k + q + 1)} \ T_{q \pm 1}^{(k)} \]

Note the similarity between operators $T_{q}^{(k)}$ and states $|k, q\rangle$.\[\]
Example: Can form a dyadic rank 2 tensor from any two vectors \( V_i \) and \( U_i \). Let \( T_{ij} = U_i V_j = \text{Cartesian rank 2} \).

This is reducible; it can be divided into subsets which only transform among themselves under rotations:

\[
T_{ij} = U_i V_j = \frac{1}{3} U \cdot V \delta_{ij} + \frac{1}{2} (U_i V_j - U_j V_i) + \frac{1}{2} (U_i V_j + U_j V_i - \frac{1}{3} U \cdot V \delta_{ij})
\]

- multiple of identity tensor \((q=0)\)
- antisymmetric tensor \((q=1)\)
- symmetric traceless \((q=2)\)

Actually, these correspond to spherical tensors with \( q = 0, 1, 2 \).

In general, spherical tensors are nice because:

1) easier to work with in spherically-symmetric systems

2) irreducible

Writing \( U_0 = U_x, \ U_{\pm 1} = -\frac{(U_x + iU_y)}{\sqrt{2}} \), \( U_{-1} = \frac{U_x - iU_y}{\sqrt{2}} \),

\[
T^{(0)} = -\frac{1}{3} U \cdot V = \frac{1}{3} (-U_0 V_0 + U_{\pm 1} V_{\pm 1} - U_{-1} V_{-1}) \quad (q=0)
\]

\[
T^{(1)}_{\pm 1} = \frac{-i}{\sqrt{2}} \left( \vec{U} \cdot \vec{V} \right)_{\pm 1} \quad (q=\pm 1, 0, -1) \quad (\text{note antisymmetric})
\]

\[
T^{(2)}_{\pm 2} = U_{\pm 1} V_{\pm 1}
\]

\[
T^{(2)}_{\mp 1} = \frac{1}{\sqrt{2}} \left( U_{\pm 1} V_0 + U_0 V_{\pm 1} \right)
\]

\[
T^{(2)}_0 = \frac{1}{\sqrt{6}} \left( 2U_0 V_0 + U_{\pm 1} V_{\mp 1} + U_{-1} V_{+1} \right)
\]

(Note not all rank 2 tensors can be formed by multiplying vectors, however.)

Given spherical tensors \( X, Z \) of ranks \( k_1 \) and \( k_2 \), can form a product tensor of rank \( k \) by:

\[
T^{(k)}_{q} = \sum_{q_1, q_2} \left< k_1 k_2 ; q_1 q_2 | k_1 k_2 ; k q \right> X^{(k_1)}_{q_1} Z^{(k_2)}_{q_2}
\]

Clebsch-Gordan coefficients like \( \left< 1 \, 1_2 ; 1_1 1_2 \right. \cdot 1_1 1_2 \)}
Proof: show that the given \( T_{q}^{(k)} \) satisfies the defining transformation law:

\[
\mathcal{D}^\dagger(R) \ T_{q}^{(k)} \mathcal{D}(R) = \sum_{q_1, q_2} \langle k, k_2, q_2 | k, k_2, q_2 \rangle \mathcal{D}^\dagger(R) \ X_{q_1}^{(k)} \mathcal{D}(R) \mathcal{D}^\dagger(R) \ Z_{q_2}^{k_2} \mathcal{D}(R)
\]

\[
= \sum_{q_1, q_2} \sum_{q_1', q_2'} \langle k, k_2, q_2 | k, k_2, q_2 \rangle \ X_{q_1'}^{(k)} \mathcal{D}(R^{-1}) \ Z_{q_2'}^{k_2} \mathcal{D}(R^{-1})
\]

Now recall the Clebsch–Gordan series: (Sakurai 3.7.69)

\[
\mathcal{D}^{(i_1)}_{m_1, m_1'} \mathcal{D}^{(i_2)}_{m_2, m_2'} = \sum_{j, m} \sum_{m_1} \langle j_1, j_2, m_1, m_2 | j_1, j_2, m \rangle \langle j_1, j_2, m | j_1, j_2, m \rangle \mathcal{D}_{m m}^{(i_1)}(R)
\]

where \( j \) runs from \( j_1 - j_2 \) to \( j_1 + j_2 \).

Use this with \( (j_1, j_2, m_1, m_2, m_2', j, m, m') \rightarrow (k, k_2, q_1, q_1, q_2, q_2, k'', q', q'') \):

\[
\mathcal{D}^\dagger(R) \ T_{q}^{(k)} \mathcal{D}(R) = \sum_{k, q''} \left( \sum_{q_1, q_2} \langle k, k_2, q_2 | k, k_2, q_2 \rangle \langle k, k_2, q_2, q_2 | k, k_2, q'' \rangle \mathcal{D}_{q_2}^{k''} \langle k, k_2, q'' | k, k_2, q'' \rangle \mathcal{D}_{q_2}^{(k)}(R^{-1}) \right)
\]

\[
= \delta_{kk''} \delta_{q_1 q_2''} \quad (C-G orthornormality; Sakurai 3.7.42)
\]

\[
= \sum_{q_1} \sum_{q_2} \langle k, k_2, q_2 | k, k_2, q_2 \rangle \ X_{q_1}^{(k)} \mathcal{D}(R^{-1}) \mathcal{D}(R^{-1}) \mathcal{D}^\dagger(R) \mathcal{D}^\dagger(R) \ Z_{q_2}^{k_2} \mathcal{D}(R)
\]

\[
= \sum_{q_1} \ T_{q_1}^{(k)} \mathcal{D}^\dagger(R^{-1}) \quad = \text{defining transformation law under rotations}
\]

Note: same C-G coefficients appear as for addition of angular momentum.

Spherical tensors are "eigen-operators of angular momentum."

\[
T_{q}^{(k)} \leftrightarrow l m
\]

\[
k \leftrightarrow l
\]

\[
q \leftrightarrow m
\]
Selection rules for spherical tensor matrix elements.

Consider states $|\alpha, j, m\rangle$ that are eigenstates of $\hat{H}, \hat{J}^2, \hat{J}_z$.

Other eigenvalues, for example $n$ in the hydrogen atom.

Then: $\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle = 0$ unless $|j - k| \leq j' < j + k$

and $m' = q + m$.

Comments: 1) These are necessary, not sufficient conditions for non-vanishing.

2) Note these are the same conditions for addition of angular momentum $(j, m)$ and $(k, q)$ to give $(j', m')$.

More generally, the Wigner–Eckhart Theorem says:

$$\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle = \frac{\langle j k | m q \mid j k; j', m' \rangle}{\sqrt{2j+1}} \frac{1}{\text{Clebsch-Gordan coefficient}}$$

This defines the reduced matrix element, which is independent of $m, m'$, and $q$. (Otherwise, the theorem would be content-free.)

So one can evaluate the reduced matrix element for some particularly convenient choice of $m, m', q$, and use that together with the C-G coefficients to get $\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle$ for any $m, m', q$.

The C-G coefficients are pure geometry (how are the states and operators oriented with respect to coordinates).

The reduced matrix element contains all the dynamics specific to the tensor operator $T_q^{(k)}$.

The selection rules now follow from those of the C-G coefficients.

Idea of proof: matrix elements of $T_q^{(k)}$ satisfy same recursion relations as C-G coefficients, so they must be proportional.
Proof: \[ \langle \alpha', j', m' | [J_+ , T_{2(1)}^{(k)} ] | \alpha , j , m \rangle = k \sqrt{(k+q)(k+q+1)} \langle \alpha', j', m' | T_{2+1}^{(k)} | \alpha , j , m \rangle \]

\[ \sqrt{(j'+m')(j'+m'+1)} \langle \alpha', j', m' = 1 | T_{2}^{(k)} | \alpha , j , m \rangle \]

\[ - \sqrt{(j+m)(j+m+1)} \langle \alpha', j', m' | T_{2}^{(k)} | \alpha , j , m \rangle \]

This has the same form as recursion relations for \[ \langle j'k' ; m'q' | jk ; j' \rangle \]. Both are first-order linear homogeneous equations with the same coefficients.

If \[ \sum_i a_{ij} x_j = 0 \] and \[ \sum_j a_{ij} y_j = 0 \], then \[ x_j = \frac{c_y}{c_x} \] independent of \( j \).

So \[ \langle \alpha', j', m' | T_{2}^{(k)} | \alpha , j , m \rangle = \text{constant independent of } \langle j'k' ; m'q' | jk ; j' \rangle \]

\[ \text{call this } \langle \alpha' \parallel T^{(k)} \parallel \alpha_j \rangle / (2j+1) \]

Example: Scalar operator (rank \( k=0 \)) \( S \):

\[ \langle \alpha' , j', m' | S | \alpha , j , m \rangle = \delta_{jj'} \delta_{mm'} \frac{1}{\sqrt{2j+1}} \langle \alpha' \parallel S \parallel \alpha_j \rangle \]

evaluate once for \( j=m=m' \), gives result for any \( m \).

Example: Vector operator (rank \( k=1 \)) \( V_0, V_+, V_- \).

Then \[ \langle \alpha', j', m' | V_q | \alpha , j , m \rangle \neq 0 \] requires:

\[ \Delta j = j' - j = 0 \text{ or } \pm 1, \quad \text{and} \quad \Delta m = m' - m = q = 0 \text{ or } \pm 1, \quad \text{and} \]

\( j, j' \) not both \( 0 \). (The \( 0 \rightarrow 0 \) transition for \( j \) is forbidden).

This applies for electric dipole radiation, for example, with \( V=R \) position operator \( R_0 = Z, \ R_+ = -\frac{1}{\sqrt{2}} (X+iy), \ R_- = \frac{1}{\sqrt{2}} (X-iy) \).

Also, for \( j \neq j' = 0 \), one can evaluate the reduced matrix element explicitly:

\[ \langle \alpha', j , m' | V_q | \alpha , j , m \rangle = \frac{\langle \alpha' , j , m | \overline{J} \cdot \overline{V} | \alpha , j , m \rangle}{k^2 j'(j+1)} \]

with \( J_0 = J_2 \) and \( J_{\pm} = \frac{1}{\sqrt{2}} (J_x \mp iJ_y) = \pm \frac{1}{\sqrt{2}} J_{\pm} \). (Proof in Sakurai, p. 241.)