What is a "path"?

A path is a function \( \vec{r}(t) \) defined for \( t' \leq t \leq t'' \):

![Diagram of a path with arrows indicating direction and continuity]

What is a sum over paths?

Partition interval \((t', t'')\) into \(N-1\) equal steps of time \(\epsilon\). Then \( t_j = t' + j\epsilon(t'' - t') \) for \( j = 0, 1, \ldots, N \). Now define values \( \vec{r}_j \) at each \( t_j \). Then:

\[
S[\text{path}] = \sum_{j=1}^{N} e \left[ \frac{m}{2} \left( \frac{|\vec{r}_j - \vec{r}_{j-1}|^2}{\epsilon} - V(\vec{r}_j) \right) \right]
\]

= action for trajectory with defined values connected by straight lines.

Now define:

\[
\langle \vec{r}''(t''|\vec{r'}, t') \rangle = \sum_{\text{paths}} e^{iS/\hbar} = \lim_{N \to \infty} \left( \frac{\hbar}{A} \right)^N \left( \frac{\hbar}{A} \right)^{N-1} e^{iS(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)/\hbar}
\]

\[
A = \text{normalization factor to be found.}
\]

Let's see if this is equivalent to Schrodinger's equation.

After one step forward \( t'' = t' + \epsilon \) with \( \epsilon \) small:

\[
\Psi(\vec{r}, t + \epsilon) = \int \frac{d^3\vec{r}_i}{A} e^{iS(\vec{r}, \vec{r}_i)/\hbar} \Psi(\vec{r}_i, t)
\]

(\(A\))

Now expand both sides in small \( \epsilon \):

\[
\Psi(\vec{r}, t + \epsilon) = \Psi(\vec{r}, t) + \epsilon \frac{\partial}{\partial t} \Psi(\vec{r}, t) + \frac{\epsilon^2}{2} \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) + \ldots
\]

For RHS:

\[
S(\vec{r}, \vec{r}_i) = \epsilon \left( \frac{m}{2} \frac{|\vec{r} - \vec{r}_i|^2}{\epsilon^2} - V(\vec{r}) \right) = \text{action for } t' \text{ to } t' + \epsilon.
\]

So RHS = \[
\int \frac{d^3\vec{r}_i}{A} \exp \left[ i \frac{\hbar}{\epsilon} \left( \frac{m}{2} \frac{|\vec{r} - \vec{r}_i|^2}{\epsilon^2} - V(\vec{r}) \right) \right] \Psi(\vec{r}_i, t)
\]
RHS of $\Psi = \left[ 1 - \frac{i \varepsilon}{\hbar} V(\vec{r}) \right] \left( \frac{d^3 \hat{r}^2}{A} \right) \exp \left[ \frac{i}{\hbar} \frac{m}{2 \varepsilon} (\vec{r} - \vec{r}_1)^2 \right] \Psi(\vec{r}_1, t)

As $\varepsilon \to 0$, there will be almost complete cancellation from the rapidly varying phase $e^{i \text{(stuff)} \varepsilon}$. The only region that contributes as $\varepsilon \to 0$ is where (stuff) $\approx 0$, or $|\vec{r} - \vec{r}_1|^2$ small.

Shift integration variable $\vec{r}_1 \to \vec{r}_1 + \vec{r}$:

RHS of $\Psi = \left[ 1 - \frac{i \varepsilon}{\hbar} V(\vec{r}) \right] \left( \frac{d^3 \hat{r}^2}{A} \right) \exp \left[ \frac{i}{\hbar} \frac{m}{2 \varepsilon} (\vec{r}_1 - \vec{r})^2 \right] \Psi(\vec{r} + \vec{r}_1, t)

Since only small $r_1$ contributes, expand:

$\Psi(\vec{r} + \vec{r}_1, t) = \Psi(\vec{r}, t) + \vec{r}_1 \cdot \hat{\nabla} \Psi(\vec{r}, t) + \frac{(\vec{r}_1 \cdot \hat{\nabla})^2}{2} \Psi(\vec{r}, t) + \ldots$

RHS of $\Psi = \left[ 1 - \frac{i \varepsilon}{\hbar} V(\vec{r}) \right] \left( \frac{d^3 \hat{r}^2}{A} \right) \exp \left[ \frac{i}{\hbar} \frac{m}{2 \varepsilon} (\vec{r}_1 - \vec{r})^2 \right] \left[ \Psi(\vec{r}, t) + \frac{\vec{r}_1 \cdot \hat{\nabla} \Psi}{\Psi} + \frac{1}{2} \left( \frac{\vec{r}_1 \cdot \hat{\nabla}}{\Psi} \right)^2 \Psi + \ldots \right]

So, putting together $\Psi$, expanded to order $\varepsilon$:

$\Psi(\vec{r}, t) + \varepsilon \frac{\partial }{\partial t} \Psi(\vec{r}, t) = \left[ 1 - \frac{i \varepsilon}{\hbar} V(\vec{r}) \right] \left( \frac{d^3 \hat{r}^2}{A} \right) \exp \left[ \frac{i}{\hbar} \frac{m}{2 \varepsilon} (\vec{r}_1 - \vec{r})^2 \right] \left[ \Psi(\vec{r}, t) + \frac{\vec{r}_1 \cdot \hat{\nabla} \Psi}{\Psi} + \frac{1}{6} \left( \frac{\vec{r}_1 \cdot \hat{\nabla}}{\Psi} \right)^2 \Psi + \ldots \right]\n
Now equate powers of $\varepsilon$:

$\varepsilon^0$

$\Psi(\vec{r}, t) = \int \frac{d^3 \hat{r}^2}{A} e^{imr_1^2/2\varepsilon} \Psi(\vec{r}, t)$

$A = \int \frac{d^3 \hat{r}^2}{A} e^{imr_1^2/2\varepsilon} = 4\pi \int_0^\infty r_1^2 dr_1 e^{imr_1^2/2\varepsilon} = \pi^{3/2} \left( \frac{2\varepsilon k \varepsilon}{m} \right)^{3/2}$

$A = (\frac{2\pi k \varepsilon}{m})^{3/2}$

$\varepsilon^1$

$\frac{\partial }{\partial t} \Psi = -\frac{i}{\hbar} V(\vec{r}) \Psi - \left( \frac{d^3 \hat{r}^2}{A} \right) e^{imr_1^2/2\varepsilon} \frac{r_1^2}{2} \hat{\nabla}^2 \Psi$

$\frac{1}{\varepsilon A} \int_0^\infty r_1^4 e^{imr_1^2/2\varepsilon} dr_1 = \frac{i \varepsilon}{2m}$
Multiply by $i\hbar$:

$$\frac{d}{dt} \psi = -\frac{\hbar}{2m} \nabla^2 \psi + V(q) \psi \quad \text{Schrodinger's equation},$$

So Feynman's path integral $\Leftrightarrow$ Schrodinger QM. (Replace $\mathbf{r}$ by $q$ = general coordinate(s)).

**Classical Limit**

$$\langle q'', t'' | q', t' \rangle = \sum_{\text{paths}} e^{iS[q(t)]/\hbar}$$

As $\hbar \to 0$, $S/\hbar$ is huge, rapidly varying $\Rightarrow$ cancellation.

The main contribution is from stationary paths where $S[q(t)]$ is slowly varying.

**What is a stationary path?**

If we change $q(t) \to q(t) + \delta q(t)$, then $S \to S + \delta S$.

A stationary path has $\delta S = 0$ for any $\delta q(t)$.

*Dominates* for $\hbar \to 0$.

In QM, must sum over all paths.

In Classical, just go on ONE path, the stationary path $q(t)$ for which $
\frac{\delta S[q(t)]}{\delta q(t)} = 0$ (this is called a functional derivative).

Only one path is followed for $\hbar \to 0$. Which one?

**Derive Classical equations of motion from $\hbar \to 0$**

$$S[q(t)] = \int_{t'}^{t''} dt \ L(q, \dot{q}, t).$$

Make a change of path $q(t) \to q(t) + \delta q(t)$ with boundary conditions $\delta q(t') = \delta q(t'') = 0$. Then:

$$\delta S = \int_{t'}^{t''} dt \left[ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

Now $\delta \dot{q} = \frac{d}{dt} \delta q(t)$, so...
\[ SS = \int_{t'}^{t''} dt \left[ \frac{\partial L}{\partial \dot{q}} \delta q + \frac{\partial L}{\partial q} \frac{d}{dt} (\delta q) \right] . \]

Integrate last term by parts...

\[ = \int_{t'}^{t''} dt \left[ \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right] + \left. \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) \right|_{t'}^{t''} \]

\[ = 0 \quad \text{because of fixed endpoints} \quad \delta q(t') = \delta q(t'') = 0. \]

In order to have \( SS = 0 \) for all \( \delta q \), need

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \]

for paths that dominate as \( t \to 0 \).

This is Lagrange’s Equations of Motion for Classical mechanics. So we have derived classical mechanics from the \( t \to 0 \) limit of Feynman’s formulation of quantum mechanics.

More generally, if there are multiple coordinates \( q_i \), then one can show as above that

\[ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \]

for each \( i \).

**3-d Quantum Mechanics:** \( q_1, q_2, q_3 \to \vec{r} \).

**Quantum Field Theory:** \( q_i \to \phi(\vec{r}) = \text{a dynamical variable} \)

for each space point.

Here \( \vec{r} \) is a label, like \( i \), not an operator.

But that’s another story.