PHYS 661 Homework Set 4 Due Thursday, February 15, 2024

Reading assignment: sections 14.4, 15.1, and 15.2 of the text.

Problem 1. Consider a particle of mass μ moving in the 3-dimensional isotropic harmonic oscillator potential

$$V(R) = \frac{1}{2}\mu\omega^2 R^2,$$

where R is the radial coordinate operator. Instead of solving exactly as we did in section 9.5 of the notes, in this problem we will use the variational principle with a trial wavefunction with radial dependence proportional to $r^n e^{-kr}$, where n and k are variational parameters. This means that the full trial wavefunction in spherical coordinates is:

$$\psi_{l,m}(n,k) = r^n e^{-kr}\, Y_l^m(\theta,\phi).$$

Here, l, m are fixed angular momentum quantum numbers associated with the operators L^2 and L_z, respectively. We could take n to vary continuously, but we will consider it to only take on integer values in this problem.

(a) Compute the energy function $E(n,k)$ for the trial wavefunction above labeled by l, m. [Hints: the answer cannot depend on the magnetic quantum number m. You do *not* want to use the explicit form of the spherical harmonics. You should get an answer of the form $E(n,k) = \frac{\hbar^2 k^2}{2m} A + \frac{m\omega^2}{k^2} B$, where A is a quadratic polynomial in l, n divided by a quadratic polynomial in n, and B is another quadratic polynomial in n. If you wish, you can send me what you got for A and B by email, and I will confirm or deny it for you before you proceed to the next parts.]

(b) Minimize the result $E(n,k)$ with respect to k, to obtain $E(n,k_{\rm min})$. Your answer should depend only on $\hbar$, ω, l, and n. [Hint: work in terms of the symbols A and B, and simplify them that way, and only plug in what they are as the very last step.]

(c) Estimate the ground state energy for $l = 0$. Which integer n does the best job? (Plug in $n = 0, 1, 2, 3, 4, 5, 6$ to find out.) How does it compare to the exact answer?

(d) Estimate the energy of the lowest state with $l = 1$. Which integer n does the best job? (Plug in $n = 0, 1, 2, 3, 4, 5, 6$.) How does it compare to the exact answer?

(e) Estimate the energy of the lowest state with $l = 2$. Which integer n does the best job? (Plug in $n = 0, 1, 2, 3, 4, 5, 6$.) How does it compare to the exact answer?

Problem 2. The Darwin term for an electron in the presence of an electric field $\vec{E}$ is given in general by: $H_D = \frac{e\hbar^2}{8m_e^2c^2}\vec{\nabla}\cdot\vec{E}$. For a point source charge $+e$, like the idealized proton in the hydrogen atom, one has $\vec{\nabla} \cdot \vec{E} = 4\pi e \delta^{(3)}(\vec{r})$, so

$$H_D = \frac{\pi e^2 \hbar^2}{2 m_e^2 c^2} \delta^{(3)}(\vec{r})$$

in the position space representation.

(a) Consider the normalized Hydrogen atom stationary state wavefunctions, which are given by $\psi_{n,l,m}(\vec{r}) = R_{n,l}(r) Y_l^m(\theta, \phi)$. For general n, l, m, compute the probability density for the electron to be at the origin, $|\psi_{n,l,m}(0)|^2$. Use eq. (10.1.41) in the text.

(b) Use your answer to part (a) to find (showing your work) that the resulting energy shifts for the $|n, l, m\rangle$ states can be written as

$$\Delta E_D = \delta_{l,0} \frac{\alpha^2}{n^3} \left(\frac{e^2}{2a_0} \right).$$

(c) How big are the resulting Darwin term energy shifts, numerically in eV, for the $n = 1$ ground state and the $n = 2$ first excited state and the $n = 3$ second excited state of the hydrogen atom? How does the result scale with the nuclear charge Z, for hydrogen-like ions?

Problem 3. It can be shown, but you don't need to do it, that the expectation values of powers of the radial coordinate, $\langle R^q \rangle$, in the hydrogen atom stationary states $|n, l, m\rangle$ obey the recursion relation:

$$\frac{q+1}{n^2} \langle R^q \rangle - (2q+1) a_0 \langle R^{q-1} \rangle + \frac{q}{4} \left[(2l+1)^2 - q^2 \right] a_0^2 \langle R^{q-2} \rangle = 0,$$

where a_0 is the Bohr radius. (This is called the Kramers–Pasternack formula, and the proof, which is not trivial, is outlined on page 215 of the text.) Use this to compute and tabulate the expectation values, $\langle n, l, m | R^q | n, l, m \rangle$, for all integers $-3 \leq q \leq 3$. As seeds, you may use the obvious result $\langle 1 \rangle = 1$ and also $\langle 1/R^2 \rangle = 2/(a_0^2 n^3 (2l+1))$, which will be derived in class using another trick. For which case or cases does the answer diverge?