Problem 1. What are the solutions of
\[
\frac{d^2 y}{dx^2} + \frac{2}{x} \frac{dy}{dx} - \frac{n(n+1)}{x^2} y = \delta(x - x')
\]
\[
\frac{d^2 y}{dx^2} + \frac{2}{x} \frac{dy}{dx} - \frac{n(n+1)}{x^2} y = f(x)
\]
on the interval $0 < x < \infty$, subject to the boundary conditions $y(0) = y(\infty) = 0$? Assume $n$ is a positive integer and $0 < x' < \infty$, and $f(x)$ is an arbitrary function.

Problem 2. The solution of
\[
y'' + \omega^2 y = g(x) \quad (0 \leq x \leq 2\pi),
\]
with arbitrary $\omega$, subject to the boundary conditions
\[
y'(0) = 0, \quad y(2\pi) = 0,
\]
can be written in the form:
\[
y(x) = \int_0^{2\pi} G(x, x', \omega) g(x') dx'.
\]
Find TWO expressions for $G(x, x', \omega)$, one involving an infinite sum over eigenfunctions, and another not involving an infinite sum but using $x_\min \equiv \min(x, x')$ and $x_\max \equiv \max(x, x')$.

[Note that the boundary conditions involve the derivative of $y$ at the origin, and $y$ at $2\pi$. Physically, this corresponds to a string with one end clamped down at $2\pi$ and the other end allowed to slide frictionlessly along the $y$ direction at $x = 0$.]

Problem 3. Consider the differential equation:
\[
\left[ (1 - x^2) \frac{d^2}{dx^2} - ax \frac{d}{dx} + n(n + a - 1) \right] u_n(x) = 0.
\]
(a) Transform this equation into the Sturm-Liouville form, and identify the functions $p(x)$, $q(x)$, and $w(x)$ and also the eigenvalue $\lambda_n$.
(b) Show that the eigenfunctions $u_n(x)$ are orthogonal for different $n$. Specify the interval of integration and the weighting factor.
(If necessary, you may assume that your solutions are polynomials, but you do not need to find them explicitly.)