Problem 1. Write each of the following functions in terms of spherical harmonics and the spherical coordinate $r$:

(a) $f(x, y, z) = y$, (b) $f(x, y, z) = xy$, (c) $f(x, y, z) = x^2 - y^2$, (d) $f(x, y, z) = x^3$.

Problem 2. In Quantum Mechanics, the angular momentum raising and lowering operators can be represented as differential operators,

\[
L_+ = L_x + iL_y = \hbar e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right),
\]
\[
L_- = L_x - iL_y = -\hbar e^{-i\phi} \left( \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \phi} \right).
\]

Show that, acting on spherical harmonics,

\[
L_+ Y^m_{\ell}(\theta, \phi) = \hbar \sqrt{(\ell - m)(\ell + m + 1)} Y^{m+1}_{\ell}(\theta, \phi),
\]
\[
L_- Y^m_{\ell}(\theta, \phi) = -\hbar \sqrt{(\ell + m)(\ell - m + 1)} Y^{m-1}_{\ell}(\theta, \phi).
\]

Problem 3. A hydrogen electron in a 2p orbital has a charge distribution:

\[
\rho(r', \theta', \phi') = \frac{q_e}{64\pi a_0^5} r'^2 e^{-r'/a_0} \sin^2 \theta'
\]

where $a_0$ is the Bohr radius. Use the Green’s function method to find the electrostatic potential $V(r, \theta, \phi)$ corresponding to this charge distribution.

[Hint: Write the charge density in terms of $Y^m_{\ell}(\theta', \phi')$. You can then do the angular part of the integral, over $\theta', \phi'$, using the orthogonality of spherical harmonics. Notice that the radial integrals split up into two parts because of the $r_<$, $r_>$ issue. You may write the result (specifically the part coming from the $r'$ radial integrals) in terms of the incomplete gamma functions:

\[
\Gamma(a, x) \equiv \int_x^\infty e^{-t}t^{a-1}dt,
\]
\[
\gamma(a, x) \equiv \int_0^x e^{-t}t^{a-1}dt,
\]

and in terms of spherical harmonics of $\theta, \phi$.]