Problem 1. (a) What linear homogeneous second-order differential equation has
\[ x^A J_{\pm m}(Bx^C) \]
as its solutions? Show that it can be written in the form:
\[ y'' + \frac{k_1}{x} y' + \left[ k_2 x^{k_3} + \frac{k_4}{x^2} \right] y = 0, \]
where \( k_1, k_2, k_3, \) and \( k_4 \) are constants that you will evaluate in terms of \( A, B, C, \) and \( m. \)

(b) Use your answer to part (a) to find the general solutions to each of:
\[ y'' + 4 x^2 y = 0, \]
\[ y'' - \frac{1}{x} y + (4 + \frac{1}{x^2}) y = 0, \]
\[ 4xy'' + 2y' + y = 0. \]

Problem 2. A cylindrical metal cavity has a radius \( a \) and a height \( L. \) The ends, \( z = 0, L, \) are at zero electrical potential. The cylindrical walls, \( r = a, \) are at a potential given by \( V(\phi, z), \) which you may assume obeys \( V(\phi, z) = -V(-\phi, z). \) Inside the cylinder is a vacuum.
(a) Show that the electrical potential everywhere inside the cylinder can be written in the form
\[ V(r, \phi, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} I_m(k_n r) \sin(k_n z) \sin(m\phi), \]
where the \( a_{mn} \) are coefficients, and \( k_n = n\pi/L, \) and \( I_m(x) \) are the modified Bessel functions.
(b) Use the boundary conditions, and the orthogonality of the sine functions, to show that the coefficients \( a_{mn} \) are given by:
\[ a_{mn} = \frac{2}{\pi L I_m(k_n a)} \int_0^{2\pi} d\phi \int_0^L dz V(\phi, z) \sin(k_n z) \sin(m\phi) \]