Physics 485 **Homework 10** **Due April 24, 2003**

Problem 1. Consider the differential equation:

$$(x^4 + 2x^2)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6x^2 y = 0.$$

Is the point $x = 0$ an ordinary point, a regular singular point, or an irregular singular point, of this differential equation? Apply the method of Frobenius. What is the indicial equation, and what is the recursion relation for coefficients? Show that the general solution is a linear combination of functions with the series solutions:

$$\begin{aligned} y_1(x) &= 1 + a_1 x^2 + \ldots \\ y_2(x) &= \frac{1}{\sqrt{x}}\left(1 + b_1 x^2 + \ldots\right), \end{aligned}$$

and find a_1 and b_1.

Problem 2. Show that the differential equation

$$y'' - xy = 0$$

(sometimes known as Airy's equation) has two linearly independent solutions when expanded around $x = 0$. Evaluate them explicitly as series, by the method of Frobenius.

Problem 3. Prove the following sum rules for Bessel's functions:

(a) $J_0(x) + 2J_2(x) + 2J_4(x) + 2J_6(x) + \ldots = 1$

(b) $J_0(x) - 2J_2(x) + 2J_4(x) - 2J_6(x) + \ldots = \cos(x)$

(c) $J_1(x) - J_3(x) + J_5(x) - J_7(x) + \ldots = \frac{1}{2}\sin(x)$

(d) $J_1(x) + 3J_3(x) + 5J_5(x) + 7J_7(x) + \ldots = \frac{x}{N}$

(e) $e^{ikr\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\theta}$

(f) $|J_0(x)|^2 + 2|J_1(x)|^2 + 2|J_2(x)|^2 + 2|J_3(x)|^2 + \ldots = 1,$

where N is a positive integer that you will find. [Hint: for part (f) consider the product $g(x,t)g(-x,t)$, where $g(x,t)$ is the generating function.]

Note that (f) proves the important result that the Bessel functions are bounded in magnitude: $|J_0(x)| \leq 1$, and $|J_m(x)| \leq 1/\sqrt{2}$. Part (e) is also important; it is an expansion of a plane wave as an infinite series of cylindrical waves.