Problem 1. Consider the infinite series:

\[ \sum_{n=3}^{\infty} \frac{1}{n \ln(n) [\ln(\ln(n))]^p} \]

For what values of \( p \) does this series converge?

Hint: Make use of the fact that

\[ \frac{d}{dx} \left( \frac{[\ln(\ln(x))]^{1-p}}{x \ln(x) [\ln(\ln(x))]^p} \right) = \frac{1 - p}{x \ln(x) [\ln(\ln(x))]^p}. \]

Problem 2. According to the theory of Special Relativity, if a particle with mass \( M \) starting at rest at \( x = 0 \) at time \( t = 0 \) is acted on by a constant force \( F \), then its displacement at time \( t \) is given by:

\[ x = \frac{Mc^2}{F} \left[ \left( 1 + \frac{F^2t^2}{M^2c^2} \right)^{1/2} - 1 \right], \]

where \( c \) is the speed of light. Find the resulting displacement \( x \) as a power series in \( t \), up to and including terms of order \( t^5 \). Compare your answer with the Newtonian (non-relativistic) result.

Problem 3. Expand \( \sin(x) \) in a Taylor series about the point \( x = \pi/4 \). Keep terms up to and including \( (x - \pi/4)^3 \).

Problem 4. The dilogarithm function (also known as the Spence function) is denoted \( \text{Li}_2(x) \). It appears in high-energy physics and statistical mechanics calculations. It can be defined as:

\[ \text{Li}_2(x) = - \int_0^x \frac{dt}{t} \ln(1 - t). \]

(a) Expand the above expression in an infinite power series in \( x \), of the form \( \sum_{n=1}^{\infty} c_n x^n \). (Give a general expression for the \( c_n \) in terms of \( n \), not just the first few terms.) This infinite series is an alternative definition for the dilogarithm.

(b) Use an appropriate convergence test to determine: for what real values of \( x \) is your answer to part (a) absolutely convergent?
(c) Compute the integral

\[ I(a, x) = \int_0^1 \frac{dt}{t} \ln(1 - atx), \]

as an infinite power series in \( x \). By comparing it to your result for part (a), express it in terms of the dilogarithm function.

(d) Using your result in part (a), evaluate \( \text{Li}_2(1) \). [Hint: you may use the result for a special infinite series mentioned in class.]

(e) An approximation for the function \( \text{Li}_2\left(\frac{x}{1 + 2x}\right) \) when \( x \) is small is: \( x + c_2x^2 + c_3x^3 \). Find \( c_2 \) and \( c_3 \).