NIU Physics PhD Candidacy Exam – Spring 2018 – Classical Mechanics

You may solve all four problems, if you choose. Only the three best problem scores count towards your total score. Total points on each problem = 40. Total possible score = 120.

Problem 1. A charged particle moves in the xy plane, under the influence of a uniform magnetic field pointing along the z direction. The Hamiltonian has the form:

$$H = \frac{1}{2m}(p_x + by)^2 + \frac{1}{2m}p_y^2,$$

where b is a constant proportional to the magnetic field strength, and p_x, p_y are the canonical momenta for the rectangular coordinates x, y. You may assume that no motion takes place along the z direction.

(a) Derive Hamilton's equations of motion from this Hamiltonian. [10 points]

(b) Find three independent constants of the motion (not counting p_z, since you should be ignoring motion in the z direction). [8 points]

(c) Solve the Hamilton's equations of motion found in part (a), taking as given the initial value data at time $t = 0$: $x(0) = x_0$, $y(0) = 0$, $p_x(0) = 0$, and $p_y(0) = p_0$. [10 points]

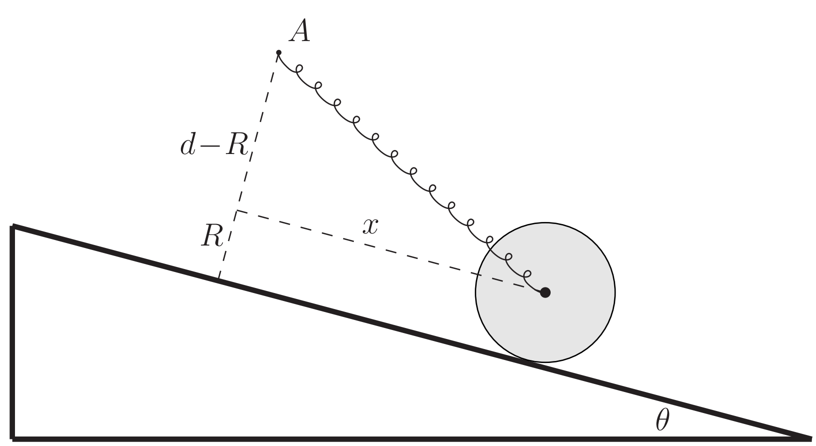
(d) Find the Lagrangian corresponding to this Hamiltonian. Derive the Euler-Lagrange equations of motion, and check that your solutions from part (c) satisfy them. [12 points]

Problem 2. A solid homogeneous flat disk of radius R and mass M rolls without slipping on a stationary inclined plane, which makes an angle θ with respect to the horizontal, as shown. The disk is in contact with the inclined plane at all times. A spring provides an attractive force on the disk towards a fixed point A, given in magnitude by $F = kr$, where k is a constant and r is the distance from the point A to the center of the disk. The point A is a distance d from the inclined plane. The disk is also acted on by the Earth's gravitational field with constant acceleration g downwards.

(a) Find the equilibrium position x, as measured in the picture. [12 points]

(b) Find the Lagrangian of the system, with x as the only dynamical variable. [12 points]

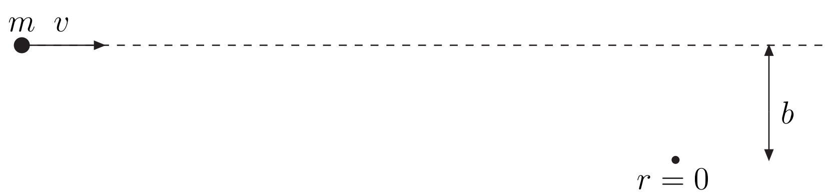
(c) Find the frequency of small oscillations around the equilibrium position. [16 points]



Problem 3. Consider a point particle of mass m moving in three dimensions in a central potential

$$V(r) = -\frac{g}{r} - \frac{k}{r^2}$$

where g and k are positive constants and r is the distance from the origin. The particle approaches from very far away with speed v and impact parameter b, as shown. (The dashed line is what the path would be if there were no potential.)

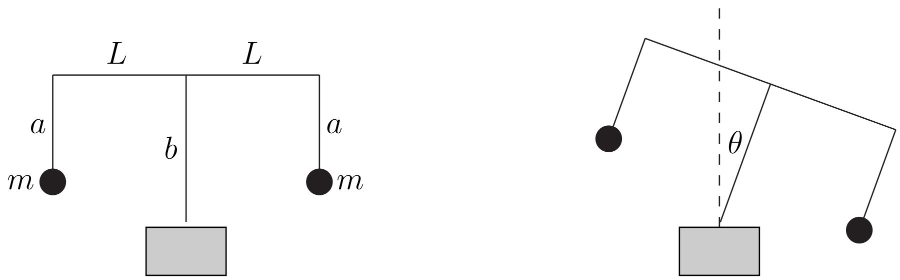


(a) Find $r_{\rm min}$, the distance of closest approach of the particle to $r = 0$. Show that the particle will go through the origin if $k > k_{\rm crit}$, where $k_{\rm crit}$ is a critical value that you will determine in terms of the other given quantities. [12 points]

In the remainder of this problem, you should assume $k < k_{\rm crit}$, and you may leave your answers in terms of $r_{\rm min}$ and the other given quantities.

(b) What is the maximum speed reached by the particle on its trajectory? [8 points]
(c) What is the maximum acceleration reached by the particle on its trajectory? [8 points]
(d) When the particle is very far from the origin again, find the angle by which it has been scattered from its original direction. You may leave your answer in terms of a single definite integral. [12 points]

Problem 4. A symmetric balancing structure below consists of two equal point masses m attached with rigid massless supports of lengths a, b, and L, connected at right angles as shown on the left below in its equilibrium balanced position. The structure is allowed to pivot on its base in one dimension (so that it always remains within the plane of the page), as measured by the angle θ from the vertical as shown on the right below. The system is in a constant gravitational field with acceleration g downwards. The base does not move.



(a) Find the Lagrangian of the system. [12 points]

(b) Under what conditions (on a, b, and/or L) is the balanced position stable with respect to small displacements? [6 points]

(c) Find the period of small oscillations. [6 points]

(d) Suppose that the condition found in part (b) is not satisfied, and the structure is displaced very slightly from its equilibrium balanced position at time $t = 0$. Find an expression for the time at which the supports of length a and b become horizontal (so that $\theta = 90$ degrees). You may leave your answer in terms of a definite integral. [16 points]