S-Plane Bode Plots - Identifying Poles and Zeros in a Circuit Transfer Function

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Abstract—A simple s-plane transformation scheme is introduced for analog circuits. Here the \( j\omega \) axis is moved step by step along the \( \sigma \) axis to meet the poles and zeros spread on the s-plane. This transformation is then applied to the circuit by adding 1) a conductance in parallel with each capacitance, and 2) a resistance in series with each inductance in the circuit. One of the major applications of the s-plane transformation is to identify and extract roots of the transfer functions. This is particularly useful in dealing with complex conjugate roots. Several examples are given that demonstrate the use and power of this technique.

Index Terms—Analog circuits, poles and zeros, s-plane transformation, roots extraction, transfer functions

I. INTRODUCTION

In an analog circuit the behavior of a transfer function is normally demonstrated by a Bode plot, i.e., the magnitude and the phase angle vs the frequency. In general, applying an input signal \( p_i(t) = \sin(\omega t) \) to a linear circuit will generate an output signal \( q_o(t) = A \sin(\omega t + \phi) \), where both the magnitude \( A \) and the phase angle \( \phi \) are frequency dependent that are displayed in a pair of Bode plots. As it turns out, in most cases these plots are sufficient to study the frequency behavior of the circuit, such as bandwidth identification. What is missing, however, is the ability to make a precise pole and zero analysis, complex frequency characterization and stability analysis of circuits for variety of input signals including exponentially decaying or increasing ac signals, as we may experience in areas such as control and signal processing. The problem with normal Bode plots is that the circuit behavior is only observed from the \( j\omega \) axis, while the poles and zeros are well spread on the entire s-plane and often far from the \( j\omega \) axis. This certainly makes the detection of poles and zeros difficult and often inaccurate. Consider a pole and a zero that are close to each other and faraway from the \( j\omega \) axis on the s-plane. Detecting the exact location of the pair in this situation is not certainly easy using a conventional Bode plot technique. The ideal case is of course when the roots (poles and zeros) fall on the \( j\omega \) axis, where a peak appears on the Bode plot when the sweeping signal crosses the root, and so it can be easily detected.

Traditionally, extracting poles and zeros of a transfer function is a two-step process. In the first step the numerator and the denominator polynomials are created, and in the second step these polynomials are solved for the roots. There are numerous tools that solve polynomials for their roots through numerical methods. These tools are widely available such as MATLAB, MATCAD and other similar mathematical packages. The problem, however, is usually with the first part. Forming circuit polynomials directly from the circuit description still remains a challenging problem [1, 3]. Within different techniques symbolic extraction of poles and zeros have gained momentum in recent years [4 - 9].

On the other hand, there has been a recent attempt to extract poles and zeros of transfer functions using Bode plots [10]. The method works fine for RC and RL circuits when the roots are on the real (\( \sigma \) axis in the s-plane. However, the method does not work for a more general case of an RLC circuit; neither is it working for the roots other than those on the \( \sigma \) axis.

Our solution to the problem here is to make the Bode plots three dimensional (plane), including the \( \sigma \) axis. This is done by moving the \( j\omega \) axis along the \( \sigma \) axis in the s-plane. In other words, implement two sweeps: have the input signal to sweep the \( j\omega \) axis, and the \( j\omega \) axis to sweep the \( \sigma \) axis. This will cover the entire complex frequency \( s = j\omega + \sigma \) plane, which results in generating the plane Bode plots. In practice, however, we cannot change the position of the \( j\omega \) axis in the s-plane, instead we can change the circuit structure so that its poles and zeros collectively move along the \( \sigma \) axis in the opposite direction.

II. MOVING THE J\omega AXIS ALONG THE \sigma AXIS

The idea is displayed in Fig. 1. Figure 1(a) shows the s-plane with a pair of poles at \( p_{1,2} = \sigma_1 \pm j\omega_1 \) locations, and Fig. 1(b) shows the \( j\omega \) axis is moved to a new location \( \sigma_1 \) on the real axis, where the poles fall on the \( j\omega \) axis. The question is then, how can we apply this transformation to the circuit so that we could get the same effect? The answer to this question is stated in Theorem 1.

There is an extension article to Reference [10] that deals with RL circuits as well as all the roots on the real access. The article is sent to ISCAS-2015 Conference for review.

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1 \( p(t) \) or \( q(t) \) can represent either voltage or current.
Theorem 1 – Given an analog circuit \( N \), consider a transfer function \( T(s) \) that has its poles and zeros spread on the s-plane. Suppose we move the \( j\omega \) axis from the origin to a location \( \sigma_i \) on the real axis to create a new s-plane, as depicted in Fig. 1. Then in order to express \( T(s) \) in the s-plane, as \( T(s') \), we need to make the following changes to the circuit \( N \).

1. Add a conductance \( \sigma_i C_i \) in parallel with each capacitance \( C_i \) in the circuit, for all \( i \), as shown in Fig. 2(a).
2. Add a resistance \( \sigma_i L_i \) in series with each inductor \( L_i \) in the circuit, for all \( i \), as shown in Fig. 2(b).

Proof: We first notice that any Laplace coefficient \( s \) in the circuit must now change to \( s' \), where \( s = s - \sigma \). On the other hand, in an RLC circuit \( s \) is always introduced into the circuit equations only through capacitors and inductors. It simply appears as \( sC \) or \( sL \). Therefore, if the transformation is applied the admittance \( sC \) becomes \( s + \sigma C \) and the impedance \( sL \) becomes \( sL + \sigma L \).

It is now clear that by repeated application of Theorem 1 we can produce multiple Bode plots, each of which representing the circuit transfer function behavior for a new s-plane, where the \( j\omega \) axis has moved to a new location on the \( \sigma \) axis. In other words, by continuously applying Theorem 1 we are now able to produce a plane of Bode plots rather than a single plot. A major application of plane Bode plots is to extract poles and zeros of a transfer function. The process is simple and is done in two steps. First, we sweep the \( j\omega \) axis along the \( \sigma \) axis until one or more poles and zeros fall on the \( j\omega \) axis. This is detected by observing the Bode plot creating peaks (up for poles and down for zeros). In the second step, the input signal is swept along the \( j\omega \) axis detecting all the poles and zeros that are present on the axis.

Our next challenge is to plot plane Bode plots by using a conventional analog circuit simulator. Here, we use WinSpice simulator, which beside the regular instructions it has features that help to get multiple plots for multiple component values.

Before we move forward, however, we must distinguish between the types of multiple plotting procedures. In a conventional situation the purpose of generating multiple (Bode) plots is to show the effect of certain component(s) variations on the frequency behavior of a transfer function, and the focus is mainly on the circuit bandwidth [11]. In other cases such as in studying root locus issues, the emphasis is on the poles and zeros that vary when a gain factor is changing. However, the multiple Bode plots here are totally of different nature. This is an s-plane transformation. The poles and zeros remain intact and they only shift horizontally on the s-plane, collectively.

Finally, it is important to know how this horizontal shift of poles and zeros must take place in order to produce a smooth and properly scaled s-plane plots. First, because the frequency axis is in log form so we need the \( \sigma \) axis to be also in log form. This causes the shifting steps to be in log form as well. Second, it is possible and in some cases necessary to apply multiple groups of steps in shifting the roots (poles and zeros). For example, to save more computational time we may adopt a stepping scheme that provides longer steps when the roots are far from the \( j\omega \) axis, and become finer and finer when the roots get close to the \( j\omega \) axis and big changes take place in the Bode plot.

The following examples show some of these features.

Example 1 – Here we consider a bridge T-Coil circuit, widely used for wideband amplifiers [11]. A simplified/reduced small signal model of the amplifier is shown in Fig. 3, and the trans-impedance \( Z_i(s) = V_i(s)/I_i(s) \) is

\[
Z_i(s) = \frac{V_i(s)}{I_i(s)} = \frac{g_{m}v_i}{\sigma_1L} + \frac{1}{\sigma_1C} + \frac{1}{sL} + \frac{1}{sC}
\]

\[
= \frac{1}{\sigma_1L} \left[ g_{m}v_i + \frac{1}{sC} + \frac{1}{sL} \right]
\]

A. Plotting plane (multiple) Bode plots

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adjustments we can equate two poles with two zeros so that the zeros are canceled by two poles and only two poles remain to deal with. The conditions for this cancellation to happen as stated in [11], are as follows:

$$C_C = 0.25C_L(1-k)/(1+k),$$

$$2C_CL(1+k)+\frac{L(1+k)}{R_i}(C_LR_i - \frac{L(1+k)}{R_i}) = C_L L$$

where,

$$L = L_a = L_o \text{, and } k = M/L$$

The parameter for this design are given as:

$$L = 50 \text{ mH}, C_L = 100 \text{ nF}, k = 0, \text{ and } R_i = 1 k\Omega.$$

For this circuit configuration we produce multiple Magnitude Bode plots for $\sigma$ ranging from 0 to -12 krad/s, which is also shown in Fig. 3. Notice that, in moving the $j\omega$ axis to the left a pair of complex conjugate poles appear on the $j\omega$ axis at $\sigma = -10 \text{ krad/s}$. The peaking frequency at this point is $f_a = 2.76 \text{ kHz}$ or $\omega_a = 17.34 \text{ krad/s}$. This will produce $\omega_0 = 20 \text{ krad/s}$, which is exactly in agreement with the theoretical value of [11 given by]:

$$\omega_0^2 = \frac{2}{LC_L(1-k)}.$$

Finally, the values of $\omega_0$ and $\sigma$ will result from the computation of the quality factor $Q = 1$, which is what it was intended in this amplifier. Also notice that the bandwidth is expanded substantially, and it is in fact doubled to $\omega_0 = 2/RCL$.

The following is the WinSpice netlist for the bridge T-Coil circuit represented for $Q = 4$.

```
** Bridge T-Coil circuit for Q = 1 ****
.control
.destroy all
.set units=degrees
.let s = s * 1.5

while s >= -4400
  let d = s - 9950
  alter @ra = d*5.0e-02
  alter @rb = d*5.0e-02
  alter @r1 = 1/(d*1.0e-07)
  alter @r2 = 1/(d*1.0e-07)
  ac dec 1000 1k 10k
  .set units=degrees
  .destroy all
.endc

.plot db(ac1.v(4)) db(ac2.v(4)) db(ac3.v(4))
.plot db(ac4.v(4)) db(ac5.v(4)) db(ac6.v(4))
.plot db(ac7.v(4)) db(ac8.v(4)) db(ac9.v(4))
.plot db(ac10.v(4)) db(ac11.v(4)) db(ac12.v(4))
 .end

.end
```

Before we leave this example we would like to make one more attempt simulating the trans-impedance of the bridge T-Coil, but this time for $Q = 4$. For this case, the circuit parameters are change to the following values [11]:

$$L = 50 \text{ mH}, L_a = 425 \text{ mH}, L_b = 425 \text{ mH}, M = -375 \text{ mH},$$

$$C_L = 100 \text{ nF}, C_C = 400 \text{ nF}, k = -15/17, \text{ and } R_i = 1 k\Omega.$$

The simulation result is shown in Fig. 4. The theoretical computation shows that there must be a pair of poles at $\omega_0 = 5 \text{ krad/s}$ or $f_0 = 796 \text{ Hz}$. This is exactly confirmed by the simulation result. Note, however, that there is also a pair of zeros that appears almost at the same frequency but with different value of $\sigma$. This is not surprising, because as we mentioned earlier, the original circuit has four poles and two zeros. In an ideal case the pair of zeros must cancel one pair of poles, leaving the other pair of poles active. However, the possibility is that, because of the component mismatch, the pair of the zeros are still shown in the simulation, as they appear in Fig. 4.

**Example 2** – Figure 5 shows a simplified and linearized schematic of a band-pass analog filter that overall has four poles and four zeros. The zeros happen to be two pairs of complex conjugates, which we would like to find their locations on the s-plane. We apply Theorem 1 by continuously changing the resistors accordingly and then simulating the circuit using WinSpice. As stated earlier, the process causes the $j\omega$ axis to move along the $\sigma$ axis until one or more poles and zeros fall on the $j\omega$ axis, in which case they are shown as, upward or downward, peaks in the Bode plots. The simulation result is graphically shown in Fig. 5, where the first pair of zeros are located at $f_{z1} = 358 \text{ Hz}$ and the second ones at $f_{z2} = 35.8 \text{ KHz}$. However, there is an important practical issue that needs to be discussed. In performing the stepwise simulation we realize that one pair of zeros appears at lower frequency range whereas the other pair is in much higher frequency range, almost two decades up. Naturally, we need smaller steps to move the $j\omega$ axis for the first pair and larger ones for
the second pair. For a smooth procedure we apply two sets of steps two decades apart. And in each the set of steps are again logarithmic to match with the frequency axis. This can be clearly observed in Fig. 5 with two different groups of log steps being applied.

In general, as we can see in Example 2, there are numerous possibilities in scanning the $\sigma$ axis in steps by the $j\omega$ axis, including linear or log steps, and also in multiple groups depending on the regions of interest on the $\sigma$ axis. This certainly provides a variety of choices for plotting transfer functions depending on the application. One of the main applications of this technique, as mentioned earlier, is to extract poles and zeros. In this situation, based on the information being available, a designer can adjust the scanning step sizes according to the location of the roots.

III. CONCLUSION

A new procedure, called s-plane Bode plots, is introduced to move the $j\omega$ axis along the real axis in steps. In each step a Bode plot (magnitude or phase angle, or both) is generated, which overall produce a plane of Bode plots. As a result of this transformation poles and zeros of transfer functions fall on the moving $j\omega$ axis allowing to identify the roots. In practice, however, instead of moving the $j\omega$ axis, some of the circuit components are going under stepwise changes to make the entire poles and zeros of the circuit functions to shift in the opposite direction on the real axis. The application of the s-plane Bode plots is not only for the extraction of circuit roots but others such as designs for bandwidth, frequency response adjustments, and circuit stability. Several examples are worked out that clearly demonstrate some applications of the proposed methodology.

REFERENCES


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4 It is of course more natural to assume log steps for the movements, but the simulator may not support it.