NIU Physics PhD Candidacy Exam - Spring 2015

Quantum Mechanics

Do Only Three Out Of Four Problems

I. FLUX QUANTUM [(5+10+20+5) PTS]

A charged particle (charge $-e$) moves in a (three dimensional) space having an infinite, impenetrable cylinder of radius $a$ along the z-axis in its center. $\psi_0$ shall be the solution of the stationary Schrödinger equation outside the cylinder without magnetic field.

a) Now we apply a magnetic field $B$ to the system, which is determined by the vector potential, $A$. Write down the corresponding Schrödinger equation.

b) Here the vector potential will be

$$A(r, \varphi, z) = \begin{cases} \frac{1}{2} B r \hat{\varphi}, & r < a \\ \frac{2}{3} B r \hat{\varphi}, & r > a \end{cases}$$

where $\hat{\varphi}$ is the angular unit vector. Calculate the magnetic field distribution, $B$, from this vector potential.

c) Use the functional form $\psi = e^{-i\gamma \chi} \psi_0$ with

$$\chi(x) = \int_{x_0}^{x} A \cdot ds$$

for the wave function and solve the Schrödinger equation corresponding to the above vector potential. Find the constant $\gamma$.

d) For which flux $B a^2$ is the wave function $\psi$ unique?

Hint: $\nabla = \left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z} \right)$, $\nabla \times \mathbf{v} = \frac{1}{r} \begin{vmatrix} \hat{r} & \hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ v_r & rv_\varphi & v_z \end{vmatrix}$.

II. NEUTRONS IN THE BOX [(10+10+10+10) PTS]

Eight non-interacting neutrons are confined to a 3D square well of size $D = 5\, \text{fm}$ ($5 \cdot 10^{-15}\, \text{m}$) such that $V = -50\, \text{MeV}$ for $0 < x < D$, $0 < y < D$, $0 < z < D$ and $V = 0$ everywhere else.

a) How many energy levels are there in this well?

b) What is the degeneracy of each energy level?

c) What is the approximate Fermi energy for this system?

d) What is the relative probability to be in the lowest energy state to the fourth lowest energy state at $k_B T = 10\, \text{MeV}$? Just write down the ratio (do not calculate the value)

Useful constants: mass of neutron = 940 MeV/c^2, $\hbar c = 197$ MeV·fm, $hc = 1240$ MeV·fm
III. ATTRACTIVE WALL [(12+12+8+8) PTS]

Let us consider a step potential with an attractive $\delta$-function potential at the edge

$$U(x) = U\theta(x) - \frac{\hbar^2 g}{2m}\delta(x).$$

a) Calculate the wave function for $E > V$.

b) Calculate the reflection coefficient $|R|^2$ and discuss the limit $E \gg U \gg \hbar^2 g/2m$.

c) Determine the wavefunction for the bound state.

d) What is the energy of the bound state?

IV. K-CAPTURE [(13+13+14) PTS]

The $K$-capture process involves the absorption of an inner orbital electron by the nucleus, resulting in the reduction of the nuclear charge $Z$ by one unit. This process is due to the non-zero probability that an electron can be found within the volume of the nucleus. Suppose that an electron is in the $1s$ state of a Hydrogen-like potential, with wavefunction given by:

$$\psi(r) = \frac{1}{\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}$$

where $a_0$ is the Bohr radius.

a) Calculate the probability that a $1s$ electron will be found within the nucleus. Take the nuclear radius to be $R$, which you may assume is much smaller than $a_0$.

b) Assume that an electron is initially in the ground state with $Z = 2$, and a nuclear reaction abruptly changes the nuclear charge to $Z = 1$. What is the probability that the electron will be found in the ground state of the new potential after the change in nuclear charge?

c) Now assume instead that the nuclear reaction leaves the electron in a state given by the wavefunction

$$\Psi(r, \theta, \phi) = A(\sin \theta \sin \phi + \sin \theta \cos \phi + \cos \theta)re^{-r/a_0},$$

where $A$ is an appropriate normalization constant. What are the possible values that can be obtained in measurements of $L^2$ and $L_z$, and with what probabilities will these values be measured?