Problem 1. A particle of mass \( m \) and momentum \( p \) is incident from the left on a one-dimensional potential well \( V(x) \), which is non-zero only between \( x = 0 \) and \( x = a \) as shown in the figure. The energy \( E = p^2/2m \) of the incident particle is very large compared to the depth of the potential, so that you may treat the potential as small, and keep only effects that are leading order in \( V(x) \) (the Born approximation).

(a) What is \( \phi(x) \), the unperturbed wavefunction (for \( V(x) = 0 \))? [3 points]

(b) Let us write the perturbed wavefunction as:

\[
\psi(x) = \phi(x) + \int_{-\infty}^{\infty} G(x, x') V(x') \phi(x') dx' + \ldots.
\]

Show that \( G(x, x') \) then obeys the differential equation

\[
\frac{\partial^2}{\partial x^2} G(x, x') + C_1 G(x, x') = C_2 \delta(x - x')
\]

where \( C_1 \) and \( C_2 \) are positive constants that you will find. [12 points]

(c) Try a solution for \( G(x, x') \) of the form:

\[
G(x, x') = \begin{cases} 
Ae^{ik(x-x')} & \text{(for } x \geq x'), \\
Ae^{-ik(x-x')} & \text{(for } x \leq x').
\end{cases}
\]

Solve for the constants \( A \) and \( k \) in terms of \( m \) and \( p \). [12 points]

(d) Find the probability that the particle will be reflected from the well. Leave your answer in terms of a well-defined integral involving the potential \( V(x) \). [13 points]
Problem 2. Consider the Hamiltonian

\[ H = \frac{p^2}{2m} - \alpha \delta(x). \]  

(1)

Although this problem can be solved exactly, let us approach it variationally and take as a guess for our ground state a Gaussian

\[ \psi(x) = Ae^{-bx^2}. \]  

(2)

(a) Find the normalization constant \( A \). [8 points]

(b) Calculate the kinetic energy. [10 points]

(c) Calculate the potential energy (the delta function). [10 points]

(d) Find \( b \) using the variational principle. [12 points]

Problem 3. Consider the three-dimensional infinite cubical well

\[ V(x, y, z) = \begin{cases} 
0, & \text{if } 0 < x < a, 0 < y < a, 0 < z < a \\
\infty, & \text{otherwise}
\end{cases} \]

(a) Find the eigenenergies and eigenstates. [10 points]

(b) Let us now introduce a perturbation

\[ V'(x, y, z) = \begin{cases} 
V_0, & \text{if } 0 < x < a/2, 0 < y < a/2, 0 < z < a \\
0, & \text{otherwise}
\end{cases} \]

Find the matrix form of \( V' \) between the first excited states. [20 points]

(c) Calculate then new eigenenergies and eigenvectors in terms of \( a \) and \( b \). [10 points]
Problem 4. Consider a particle of mass $m$ and charge $q$ in a three-dimensional harmonic oscillator described by the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

with $p = (p_x, p_y, p_z)$ and $r = (x, y, z)$.

(a) Show that the eigenstates of $H_0$ are eigenstates of angular momentum $L_z$. [8 points]

Similarly one can show that the eigenstates of $H_0$ may also be chosen as eigenstates of angular momentum $L_z$ and $L_y$, so that the eigenstates of $H_0$ may be labeled $|n, \ell, m_z\rangle$, where $E_n = \hbar\omega(n + \frac{3}{2})$ with $n = 0, 1, 2, \ldots$ is the eigenvalue of $H_0$, $\hbar^2\ell(\ell + 1)$ is the eigenvalue of $L^2$ and $m_z\hbar$ is the eigenvalue of $L_z$. We assume that at time $t = -\infty$ the oscillator is in its ground state, $|0, 0, 0\rangle$. It is then acted upon by a spatially uniform but time dependent electric field

$$\mathbf{E}(t) = \mathbf{E}_0 \exp(-t^2/\tau^2)\hat{z}$$

(where $\mathbf{E}_0$ and $\tau$ are constant).

(b) Show that, to first order in the perturbation, the only possible excited state the oscillator could end up in is the $|1, 1, 0\rangle$ state. [8 points]

(c) What is the probability for the oscillator to be found in this excited state at time $t = \infty$? [Note that $\int_{-\infty}^{\infty} \exp[-(x - c)^2] \, dx = \sqrt{\pi}$ for any complex constant $c$.] [8 points]

(d) The probability you obtain should vanish for both $\tau \to 0$ and $\tau \to \infty$. Explain briefly why this is the case. [8 points]

(e) If instead the oscillator was in the $|1, 1, 0\rangle$ state at time $t = -\infty$, show that the probability that it ends up in the ground state at time $t = \infty$ is identical to what was found in part c). [8 points]