Problem 1.

(a) In general, what is the first order correction to the energy of a quantum state for a one dimensional system with a time independent perturbation given by $H'$? [8 points]
(b) Suppose in an infinite square well between $x = 0$ and $x = a$ the perturbation is given by raising one half of the floor of the well by $V_0$. What is the change in energy to the even and to the odd states? [16 points]
(c) Now suppose the perturbation is given by $\alpha \delta(x - a/2)$ where $\alpha$ is constant. What is the first order correction to the allowed energies for the even and odd states? [16 points]

Problem 2.

We consider scattering off a spherical potential well given by

$$V(r) = \begin{cases} 
-V_0 & r \leq a \\
0 & r > a
\end{cases} \quad V_0, a > 0$$

The particle’s mass is $m$. We restrict ourselves to low energies, where it is sufficient to consider $s$ wave scattering (angular momentum $l = 0$).

(a) Starting from the Schrödinger equation for this problem, derive the phase shift $\delta_0$. [14 points]

(b) Calculate the total scattering cross section $\sigma$ assuming a shallow potential well ($a\sqrt{2mV_0/\hbar^2} \ll 1$). [10 points]

(c) Show that the same total scattering cross section $\sigma$ as in b) is also obtained when using the Born approximation. Note: part c) is really independent of parts a) and b). [16 points]
Problem 3.

For a quantum harmonic oscillator, we have the position ˆ\(x\) and momentum ˆ\(p_x\) operators in terms of step operators

\[
\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \quad \text{and} \quad \hat{p}_x = i\sqrt{\frac{m\hbar}{2}}(a^\dagger - a)
\]  

(1)
giving a Hamiltonian  \(\hat{H} = \hbar\omega(a^\dagger a + \frac{1}{2})\).

(a) The eigenstates with energy \((n + \frac{1}{2})\hbar\omega\) in bra-ket notation are |\(n\rangle\). Express the eigenstates in terms of the step operators and the state |\(0\rangle\) (no need to derive). [8 points]

(b) Show that the eigenstates |\(1\rangle\) and |\(2\rangle\) are normalized using the fact that |\(0\rangle\) is normalized (or derive the normalization factor for those states, in case your result from (a) is not normalized). [8 points]

(c) Calculate the expectation values of \(\langle \hat{x}^2 \rangle\) and \(\langle \hat{p}_x^2 \rangle\) for the eigenstates |\(n\rangle\). [8 points]

(d) Using the result from (c), show that the harmonic oscillator satisfies Heisenberg’s uncertainty principle (consider only eigenstates). [8 points]

(e) The term \(H' = \gamma x^2\) is added to the Hamiltonian. Find the eigenenergies of \(\hat{H} + H'\). [8 points]

Problem 4.

(a) We want to study the spin-orbit coupling for an atomic level with \(l = 2\). How will this level split under the interaction \(\zeta \mathbf{L} \cdot \mathbf{S}\)? Give also the degeneracies. [8 points]

(b) Show that for an arbitrary angular momentum operator (integer and half-integer), we can write

\[
J_\pm |jm\rangle = \sqrt{(j \mp m_j)(j \pm m_j + 1)}|jm\pm 1\rangle
\]

(2)
take \(\hbar = 1\) [Hint: Rewrite \(J_\pm J_\mp\) in terms of \(J^2\) and \(J_z\)]. [10 points]

(c) Since \(m_j\) is a good quantum number for the spin-orbit coupling, we can consider the different \(m_j\) values separately. Give the matrix for \(\zeta \mathbf{L} \cdot \mathbf{S}\) in the \(|lm, \frac{1}{2}\sigma\rangle\) basis with \(\sigma = \pm \frac{1}{2}\) for \(m_j = 3/2\). Find the eigenvalues and eigenstates of this matrix. [12 points]

(d) Write down the matrix for the spin-orbit coupling in the \(|jm\rangle\) basis for \(m_j = 3/2\). [10 points]