Problem 1. In many systems, the Hamiltonian is invariant under rotations. An example is the hydrogen atom where the potential $V(r)$ in the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(r),$$

depends only on the distance to the origin. An infinitesimally small rotation along the $z$-axis of the wavefunction is given by

$$R_{z,d\phi}\psi(x, y, z) = \psi(x - yd\phi, y + xd\phi, z),$$

(a) Show that this rotation can be expressed in terms of the angular momentum component $L_z$. [10 points]

(b) Starting from the expression of $L_z$ in Cartesian coordinates, show that $L_z$ can be related to the derivative with respect to $\phi$ in spherical coordinates. Derive the $\phi$-dependent part of the wavefunction corresponding to an eigenstate of $L_z$. [10 points]

(c) Show that if $R_{z,d\phi}$ commutes with the Hamiltonian, then there exist eigenfunctions of $H$ that are also eigenfunctions of $R_{z,d\phi}$. [10 points]

(d) Using the fact that $L_i$ with $i = x, y, z$ commute with the Hamiltonian, show that $L^2$ commutes with the Hamiltonian. [10 points]

Problem 2. To a harmonic oscillator Hamiltonian

$$H = \hbar \omega a^\dagger a,$$

we add a term

$$H' = \lambda (a^\dagger + a).$$

This problem is known as the displaced harmonic oscillator. It can be diagonalized exactly by adding a constant (let us call it $\Delta$) to the step operators.

(a) Express the constant $\Delta$ in terms of $\hbar \omega$ and $\lambda$. [8 points]

(b) The energies are shifted by a constant energy. Express that energy in terms of $\hbar \omega$ and $\lambda$. [8 points]

(c) Express the new eigenstates $|\tilde{n}\rangle$ in terms of the displaced oscillator operator $\tilde{a}^\dagger$. [8 points]

(d) Calculate the matrix elements $\langle \tilde{n}|0\rangle$. [8 points]

(e) An harmonic oscillator is in the ground state of $H$. At a certain time, the Hamiltonian suddenly changes to $H + H'$. Plot the probability and change in energy for the final states $|\tilde{n}\rangle$ for $\tilde{n} = 0, \cdots, 5$ for $\Delta = 2$. [8 points]
Problem 3. We consider scattering off a spherical potential well given by

\[ V(r) = \begin{cases} 
- V_0 & r \leq a \\
0 & r > a 
\end{cases} \]

The particles’ mass is \( m \). We restrict ourselves to low energies, where it is sufficient to consider \( s \) wave scattering (angular momentum \( l = 0 \)).

(a) Starting from the Schrödinger equation for this problem, derive the phase shift \( \delta_0 \). \([14 \text{ points}]\]

(b) Calculate the total scattering cross section \( \sigma \) assuming a shallow potential well \((a \sqrt{2mV_0/\hbar^2} \ll 1)\). \([10 \text{ points}]\)

(c) Show that the same total scattering cross section \( \sigma \) as in b) is also obtained when using the Born approximation. Note: part c) is really independent of parts a) and b). \([16 \text{ points}]\)

Problem 4. The normalized wavefunctions for the 2\( s \) and 2\( p \) states of the hydrogen atom are:

\[
\psi_{2s} = \frac{1}{\sqrt{32\pi a^3}} (N - r/a) e^{-r/2a} \\
\psi_{2p,0} = \frac{1}{\sqrt{32\pi a^3}} (r/a) e^{-r/2a} \cos \theta \\
\psi_{2p,\pm1} = \frac{1}{\sqrt{64\pi a^3}} (r/a) e^{-r/2a} \sin \theta e^{\pm i\phi}.
\]

where \( a \) is the Bohr radius and \( N \) is a certain rational number.

(a) Calculate \( N \). (Show your work; no credit for just writing down the answer.) \([10 \text{ points}]\)

(b) Find an expression for the probability of finding the electron at a distance greater than \( a \) from the nucleus, if the atom is in the 2\( p, +1 \) state. (You may leave this answer in the form of a single integral over one variable.) \([10 \text{ points}]\)

(c) Now suppose the atom is perturbed by a constant uniform electric field \( \vec{E} = E_0 \hat{z} \). Find the energies of the 2\( s \) and 2\( p \) states to first order in \( E_0 \). \([20 \text{ points}]\)