Problem 1.

Consider a particle with mass \( m \) confined to a three-dimensional spherical potential well

\[
V(r) = \begin{cases} 
0, & r \leq a \\
V_0, & r > a 
\end{cases}
\]  

(1)

a) Give the Schrödinger equation for this problem.

b) Determine the explicit expressions for the ground state energy and the ground state wave function in the limit \( V_0 \rightarrow \infty \).

c) For the more general case \( 0 < V_0 < \infty \), determine the transcendental equation from which we can obtain the eigenenergies of the particle for angular momentum \( l = 0 \).

d) Which condition must be fulfilled such that the transcendental equation derived in c) can be solved? (Hint: consider a graphical solution of the equation.) Compare this result with a particle in a one-dimensional rectangular well of depth \( V_0 \).

Problem 2.

The ground state energy and Bohr radius for the Hydrogen atom are

\[
E_1 = -\frac{\hbar^2}{2am_B^2}, \quad a_B = \frac{4\pi\varepsilon_0\hbar^2}{e^2m}.
\]  

(2)

a) Calculate the ground state energy (in eV) and Bohr radius (in nm) of positronium (a hydrogen-like system consisting of an electron and a positron).

b) What is the degeneracy of the positronium ground state due to the spin? Write down the possible eigenvalues of the total spin together with the corresponding wavefunctions.

c) The ground state of positronium can decay by annihilation into photons. Calculate the energy and angular momentum released in the process and prove there must be at least two photons in the final state.
Problem 3.

A particle of mass $m$ moves in one dimension inside a box of length $L$. Use first order perturbation theory to calculate the lowest order correction to the energy levels arising from the relativistic variation of the particle mass. You can assume that the effect of relativity is small. Note that the free particle relativistic Hamiltonian is $\hat{H}_{\text{rel}} = \sqrt{\frac{p^2 c^2}{m^2} + p^2 c^2} - mc^2$.

Problem 4.

a) Prove the variational theorem that states that for any arbitrary state

$$\langle \psi | \hat{H} | \psi \rangle \geq E_0.$$  \hspace{1cm} (3)

b) Consider the Hamiltonian for a particle moving in one dimension

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_0 \left( \frac{\hat{x}}{a} \right)^6,$$  \hspace{1cm} (4)

where $m$ is mass, $a$ a length scale, and $V_0$ an energy scale. Is the wavefunction

$$\psi(x) = C(x^2 - a^2)e^{-x^2/d^4},$$  \hspace{1cm} (5)

where $C$ is a normalization constant, $d$ an adjustable parameter with dimensions of length, a good choice for the variational approximation to the ground state? Why or why not?

c) Depending on the answer of the previous part: If it is a good choice, make a rough order-of-magnitude estimate of the optimal choice for $d$. If it is not a good choice, propose a better variational wavefunction, including an estimate of the length scale.