Problem 1.

A particle moves in a 1 dimensional potential described by an attractive delta function at the origin. The potential is:

\[ V(x) = -W \delta(x) \]

(a) Discuss and determine the wavefunctions valid for bound state solutions of this system.

(b) Show that there is only one bound state and determine its energy.

Problem 2.

In this problem, \(|0\rangle, |n\rangle\) are the shorthand for the eigenstates of the 1 dimensional simple harmonic oscillator (SHO) Hamiltonian, with \(|0\rangle\), denoting the ground state. The \(\hat{a}^\dagger\) and \(\hat{a}\) are the SHO raising and lowering operators. (sometimes termed creation and destruction (annihilation) operators).

(a) Prove that the following state vector \(|z\rangle\) is an eigenstate of the lowering operator \(\hat{a}\) and that its eigenvalue is \(z\).

\[ |z\rangle = e^{z\hat{a}^\dagger}|0\rangle \]

The \(z\) is an arbitrary complex number. (Note, knowledge of the expansion of \(e^x\) will be useful. )

(b) Evaluate \((z_1|z_2\rangle\), where \(z_1\) and \(z_2\) are arbitrary complex numbers, and use this result to normalize state \(|z\rangle\).

Problem 3.

Let the potential \(V = 0\) for \(r < a_0\) (the Bohr radius) and \(V = \infty\) for \(r > a_0\). \(V\) is a function of \(r\) only.

(a) What is the energy of an electron in the lowest energy state of this potential?

(b) How does this compare to the kinetic energy of the 1s state of Hydrogen?

(c) What is the approximate energy of the lowest energy state with angular momentum greater than 0 (you can leave this result in integral form)?

Problem 4.

In a magnetic resonance experiment a specimen containing nuclei of spin \(I = \frac{1}{2}\) and magnetic moment \(\mu = h \gamma I\) is placed in a static magnetic field \(B_0\) directed along the \(z\)-axis and a field \(B_1\) which rotates in the \(xy\)-plane with angular frequency \(\omega\).
(a) Write down the Hamiltonian for the system.

(b) If the wave function is written

\[ \psi(t) = c_+(t)\chi_{\frac{1}{2}} + c_-(t)\chi_{-\frac{1}{2}} \]

where \( \chi_{\frac{1}{2}} \) and \( \chi_{-\frac{1}{2}} \) are the spin eigenfunctions, show that

\[ i\frac{dc_+}{dt} = \frac{1}{2}\omega_0 c_+ + \frac{1}{2}\omega_1 c_- e^{-i\omega t} \]

and

\[ i\frac{dc_-}{dt} = -\frac{1}{2}\omega_0 c_- + \frac{1}{2}\omega_1 c_+ e^{i\omega t} \]

and where \( \omega_0 = \gamma B_0 \) and \( \omega_1 = \gamma B_1 \). Assuming that the system starts in the state \( \chi_{-\frac{1}{2}} \), i.e. \( c_+(0) = 0 \) and \( c_-(0) = 1 \), solve these equations to show that subsequently the probability that the system is in the state \( \chi_{\frac{1}{2}} \) is

\[ |c_+|^2 = \omega_2^2 \frac{\sin^2 \frac{1}{2}[(\omega - \omega_0)^2 + \omega_1^2]^{\frac{1}{2}} t}{(\omega - \omega_0)^2 + \omega_1^2} \]