I. ELECTRON IN WELL [(12+10+18) PTS]

An electron (ignore the effects of spin) is in a 1-dimensional potential well $V(x)$ with

$$V(x) = \begin{cases} 
0 \text{ eV for } 0 < x < 0.4 \text{ nm} \text{ and } x > 10.4 \text{ nm} \\
200 \text{ eV for } x < 0 \text{ and } 0.4 \text{ nm} < x < 10.4 \text{ nm}
\end{cases}$$

a) What are the lowest two energies (in eV) for the bound states in the region with $0 < x < 0.4 \text{ nm}$? How do they compare to the energies levels of an infinite well with the same width?

b) How many bound states are in this well?

c) If an electron is initially in the well, what is the relative probability for an electron in the lowest energy state to tunnel through the barrier compared to an electron in the second lowest energy state? Give the result in a number good to a factor of 10.

Electron mass = 0.5 MeV/c$^2$; $\hbar c = 197$ eV nm; $hc = 1240$ eV nm

II. SLIGHTLY RELATIVISTIC 1D HARMONIC OSCILLATOR [(6+10+12+12) PTS]

You know that the concept of potential energy is not applicable in relativistic situations. One consequence of this is that the only fully relativistic quantum theories possible are quantum field theories. However, there do exist situations where a particle’s motion is “slightly relativistic” (e.g., $v/c \sim 0.1$) and where the force responds quickly enough to the particle’s position that the potential energy concept has approximate validity.

Here we consider the one-dimensional harmonic oscillator, defined by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + m\omega^2 \hat{x}^2.$$

Reminder: The eigenvalues of the harmonic oscillator are $E_n = \hbar \omega (n + \frac{1}{2})$ and the eigenstates can be expressed as $|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$ with the creation ($\hat{a}^\dagger$)/annihilation ($\hat{a}$) operators given by $(\hat{x}/x_0 \pm i\hat{p}/p_0)/\sqrt{2}$, respectively $[x_0 = \sqrt{\hbar/(m\omega)}$, $p_0 = \sqrt{m\hbar\omega}]$. $\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ and $\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$.

a) Using the relativistic energy relation $E = (m^2 c^4 + p^2 c^2)^{1/2}$, derive the $\hat{p}^4$ order correction to the harmonic oscillator Hamiltonian given above.

b) Calculate $\langle n | \hat{p}^4 | 0 \rangle$.

Hint: The result has the form $p_0^4 (C_0 \delta_{n,0} + C_2 \delta_{n,2} + C_4 \delta_{n,4})$, with numerical prefactors $C_i$ that you will find.

c) Calculate the leading non-vanishing energy shift of the ground state due to this relativistic perturbation.

d) Calculate the leading corrections to the ground state eigenvector $|0\rangle$. 
III. COUPLED SPINS [(18+6+16) PTS]

Consider two spatially localized spins $1/2$, $S_1$ and $S_2$, coupled by a transverse exchange interaction and in an inhomogeneous magnetic field. The Hamiltonian is:

$$H = b_1 S_{1z} + b_2 S_{2z} - k (S_1^+ S_2^- + S_1^- S_2^+),$$

where $b_1$ and $b_2$ are proportional to the magnetic fields at the two sites and $k$ measures the strength of the exchange coupling, and $S_1^\pm = S_{1x} \pm i S_{1y}$ and $S_2^\pm = S_{2x} \pm i S_{2y}$.

a) Find the energy eigenvalues.
b) If both spins are in the $+z$ directions at time $t = 0$, what is the probability that they will both be in the $+z$ direction at a later time $t$?
c) If $b_1 = b_2$, and at time $t = 0$ the spin $S_1$ is in the $+z$ direction and $S_2$ is in the $-z$ direction, then what is the probability that $S_1$ will be in the $-z$ direction at a later time $t$?

IV. SPHERICAL POTENTIAL WELL [(10+10+10+10) PTS]

Consider a spherical potential well with

$$U(r) = \begin{cases} 0 & r < a \\ \infty & r > a \end{cases}$$

a) Give the radial part of the Hamiltonian for $r < a$.
b) We are now only interested in the wavefunction for $l = 0$. Give the general solution of the differential equation by trying a solution $R = P/r$. (Normalization is not necessary).
c) Determine the behavior of the wavefunction for $r \to 0$. What does the condition that the wavefunction should not diverge at $r = 0$ imply?
d) Determine the energy eigenvalues.