I. PARTICLE IN POTENTIAL WELL [(16+12+12) PTS]

Here we consider a particle of mass \(m\) confined in a one-dimensional potential well defined by

\[
U(x) = \begin{cases} 
\alpha \delta(x), & |x| < a \\
\infty, & |x| \geq a
\end{cases}
\]

for \(a > 0\) and \(\alpha > 0\). The energy levels (eigenvalues) \(E_n\) can be calculated without perturbation theory.

a) For \(m a^2/\hbar^2 \gg 1\), show that the lowest energy levels \(E_n \sim 1\) are pairs of close lying levels.

b) Find the spectrum for large energies \(n \gg 1\).

c) Find the energy levels for \(\alpha < 0\).

II. DENSITY MATRIX [(10+10+5+15) PTS]

Let us consider a system in a (normalized) pure quantum state \(|\psi\rangle\) and define the operator

\[
\hat{\rho} \equiv |\psi\rangle \langle \psi| ,
\]

which is called the density matrix.

a) Show that the expectation value of an observable associated with the operator \(\hat{A}\) in \(|\psi\rangle\) is \(\text{Tr}(\hat{\rho} \hat{A})\).

b) Frequently physicists don’t know exactly which quantum state their system is in. (For example, silver atoms coming out of an oven are in states of definite \(\mu\) projection, but there is no way to know which state any given atom is in.) In this case there are two different sources of measurement uncertainty: first, we don’t know what state they system is in (statistical uncertainty, due to our ignorance) and second, even if we did know, we couldn’t predict the result of every measurement (quantum uncertainty, due to the way the world works). The density matrix formalism neatly handles both kinds of uncertainty at once. If the system could be in any of the states \(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_i\rangle, \ldots\) (which do not necessarily form a basis set), and if it has probability \(p_i\) of being in state \(|\psi_i\rangle\), then the density matrix

\[
\hat{\rho} = \sum_{i} p_i |\psi_i\rangle \langle \psi_i|
\]

is associate with the system. Show that that expectation value of the observable associated with \(\hat{A}\) is still given by \(\text{Tr}(\hat{\rho} \hat{A})\).

c) Calculate \(\text{Tr}(\hat{\rho})\).

d) Now we consider the example of two spin-1/2 particles. Give the density matrix for the following cases in \(|+\rangle, |-\rangle\)-basis (\(\pm\) refer to the sign of the eigenvalue of the spin operator):

- Both spins have the same orientation.
- The total spin is zero.

\textbf{Hint:} The trace can be written as \(\text{Tr}(\hat{A}) = \sum_k \langle \phi_k | \hat{A} | \phi_k \rangle\), where the \(|\phi_k\rangle\) form an orthonormal basis.
III. ELECTRON IN A HOMOGENEOUS MAGNETIC FIELD \([16+12+12)\) PTS

Let us consider electrons \((\text{mass } \mu, \text{ charge } -e)\), which move in a spatially homogeneous constant magnetic field in the \(x-y\) plane.

a) Show that the Hamiltonian can be written as

\[
\hat{H} = \frac{2}{\mu} \left[ -\hbar^2 \frac{\partial^2}{\partial \eta^2} - \hbar \frac{eB}{4c} \left( \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \bar{\eta}} \right) \right] + \left( \frac{eB}{4c} \right)^2 \eta \bar{\eta},
\]

using the coordinate transformation \(\eta = x + iy, \bar{\eta} = x - iy\).

b) Use the product ansatz \(\psi(\eta, \bar{\eta}) = f(\eta) \exp(-\eta \bar{\eta}/(4l^2))\) with \(l = \sqrt{\hbar c/(eB)}\) and construct the stationary Schrödinger equation for \(f\). Show that the wave function related to the lowest eigenvalue \(E_0 = \hbar \omega_c\left(\omega_c = eB/(\mu c)\right)\) satisfies the condition \(\partial_\eta f = 0\) and is therefore a polynomial in \(\eta\).

c) Calculate the component of the angular momentum operator \(\hat{l}_z\) perpendicular to the \(x-y\) plane in complex coordinates \(\eta\) and \(\bar{\eta}\). Show that \([\hat{H}, \hat{l}_z] = 0\) and that the functions \(\varphi_m(\eta) = \eta^m \exp(-\eta \bar{\eta}/(4l^2))\) are eigenfunctions of \(\hat{l}_z\). Calculate their eigenvalues.

Hint: The momentum operator for a charged particle \((\text{charge } q)\) in a magnetic field transforms as \(\hat{p} \rightarrow \hat{p} - \frac{e}{c} \hat{A}\) with the vector potential for a homogeneous field being \(\hat{A} = -\frac{1}{2}(r \times \hat{B})\). In order to rewrite the partial derivatives in complex coordinates, use the chain rule, e.g., express \(\frac{\partial}{\partial x} f(\eta, \bar{\eta})\) in terms of \(\frac{\partial}{\partial \eta}\) and \(\frac{\partial}{\partial \bar{\eta}}\).

IV. ANHARMONIC OSCILLATOR \([(12+12+16)\) PTS]

The Hamiltonian

\[
\hat{H} = \hat{H}_0 + \hat{V} = \left[ \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \right] + \left[ \gamma \hat{x}^4 \right]
\]

describes a one-dimensional anharmonic oscillator, where \(\hat{V}\) is a perturbation of the harmonic oscillator \(\hat{H}_0\). As we know the eigenvalues of the harmonic oscillator, \(\hat{H}_0\), are \(E_n^{(0)} = \hbar \omega \left( n + \frac{1}{2} \right)\) and the eigenstates can be expressed as \(|\psi_n^{(0)}\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |\psi_0^{(0)}\rangle\) with the creation (or raising) operator \(\hat{a}^\dagger = (\hat{x}/x_0 + i x_0 \hat{p}/\hbar)/\sqrt{2}\) [the corresponding annihilation (or lowering) operator \(\hat{a} = (\hat{x}/x_0 - i x_0 \hat{p}/\hbar)/\sqrt{2}\) for \(n = 0, 1, 2, \ldots\) \([x_0 = \sqrt{\hbar/(m \omega)}\)\]. \(\hat{a}^\dagger |\psi_0^{(0)}\rangle = \sqrt{n+1} |\psi_{n+1}^{(0)}\rangle\) and \(\hat{a} |\psi_n^{(0)}\rangle = \sqrt{n} |\psi_{n-1}^{(0)}\rangle\).

a) Assuming that \(\gamma\) is small, calculate the ground state energy, \(E_0^{(1)}\), in first-order perturbation theory.

b) Calculate the ground state eigenstate, \(|\psi_0^{(1)}\rangle\), in first-order perturbation theory.

c) Calculate the energy eigenvalues in first order perturbation theory for arbitrary \(n\), i.e., \(E_n^{(1)}\).