Problem 1

We consider a system made of the orthonormalized spin states $|+\rangle$ and $|-\rangle$, where $S_z|\pm\rangle = \pm(\hbar/2)|\pm\rangle$. Initially both of these states are energy eigenstates with the same energy $\epsilon$.

a) An interaction $V$ couples these spin states, giving rise to the matrix elements

$$\langle 1|V|1 \rangle = \langle 2|V|2 \rangle = 0 \quad \text{and} \quad \langle 1|V|2 \rangle = \Delta$$

Give the Hamiltonian $H$ of the interacting system.

b) Determine the energy eigenvalues of $H$.

c) Show that the states $|A\rangle$ and $|B\rangle$

$$|A\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle + \frac{\Delta^*}{|\Delta|}|-\rangle \right) \quad \text{and} \quad |B\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle - \frac{\Delta^*}{|\Delta|}|-\rangle \right)$$

are orthonormalized eigenstates of the interacting system.

d) Determine the time evolution of a state $|\psi(t)\rangle$ for which $|\psi(t = 0)\rangle = |+\rangle$.

e) For the state $|\psi(t)\rangle$, calculate the probability that a measurement of $S_z$ at time $t$ yields $\pm\hbar/2$.

Problem 2

A particle of mass $m$ is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and normalized wave function.
Problem 3.

Given a two-dimensional oscillator with Hamiltonian

\[ H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + kmxy, \]  

(1)

a) What is the time dependence of \( d\langle x \rangle / dt \)?

b) What is the time dependence of \( d\langle p_x \rangle / dt \)?

c) For \( k = 0 \), what are the energies of the ground state and the first and second excited states? What are the degeneracies of each state?

d) For \( k > 0 \), using first-order perturbation theory, what are the energy shifts of the ground state and the first excited states?

Problem 4.

a) We want to study the spin-orbit coupling for a level with \( l = 3 \). How do you expect that this level will split under the interaction \( \zeta \mathbf{L} \cdot \mathbf{S} \)? Give also the degeneracies.

b) Show that for an arbitrary angular momentum operator (integer and half-integer), we can write

\[ J_\pm |jm_j\rangle = \sqrt{(j \mp m_j)(j \pm m_j + 1)} |jm_j\rangle \]  

(2)

(take \( \hbar = 1 \)).

c) Since \( m_j \) is a good quantum number for the spin-orbit coupling, we can consider the different \( m_j \) values separately. Give the matrix for \( \zeta \mathbf{L} \cdot \mathbf{S} \) in the \( |lm, \frac{1}{2}\sigma\rangle \) basis with \( \sigma = \pm \frac{1}{2} \) for \( m_j = 5/2 \). Find the eigenvalues and eigenstates of this matrix.

d) Write down the matrix for the spin-orbit coupling in the \( |jm_j\rangle \) basis for \( m_j = 5/2 \).

e) Obtain the same eigenstates as in question c) by starting for the \( m_j = 7/2 \) state using the step operators.