2005 Ph.D. Qualifier, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.
A particle of mass $m$ in an infinitely deep square well extending between $x = 0$ and $x = L$ has the wavefunction

$$\Psi(x,t) = A \left[ \sin \left( \frac{\pi x}{L} \right) e^{-iE_1t/\hbar} - \frac{3}{4} \sin \left( \frac{3\pi x}{L} \right) e^{-iE_3t/\hbar} \right],$$

where $A$ is a normalization factor and $E_n = n^2\hbar^2/8mL^2$.

(a) Calculate an expression for the probability density $|\Psi(x,t)|^2$, within the well at $t = 0$.

(b) Calculate the explicit time-dependent term in the probability density for $t \neq 0$.

(c) In terms of $m$, $L$, and $\hbar$, what is the repetition period $T$ of the complete probability density?

Problem 2.
Let us consider the spherical harmonics with $l = 1$.

(a) Determine the eigenvalues for $aL_z$, where $a$ is a constant.

(b) Determine the matrix for $L_x$ for the basis set $|lm\rangle$ with $l = 1$ using the fact that

$$L_\pm |lm\rangle = h\sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle.$$  (2)

and that the step operators are given by $L_\pm = L_x \pm iL_y$.

(c) Determine the eigenvalues of $aL_x$ for the states with $l = 1$.

(d) Determine the matrix for $L^2$ from the matrices for $L_+, L_-$, and $L_z$.

Problem 3.
Consider a two-dimensional harmonic oscillator

$$H_0 = \hbar\omega_x a_x^\dagger a_x + \hbar\omega_y a_y^\dagger a_y$$  (3)

with $\hbar\omega_x \ll \hbar\omega_y$. The number of excited states is given by $N = n_x + n_y$, where $a_x^\dagger |n_x\rangle = \sqrt{n_x + 1} |n_x + 1\rangle$ and $a_y^\dagger |n_y\rangle = \sqrt{n_y + 1} |n_y + 1\rangle$.

(a) Express the normalized state with $N = 2$ with the lowest energy in terms of the step operators and the vacuum state $|0\rangle$, i.e. the state with no oscillators excited.

The system is now perturbed by

$$H_1 = K(a_x^\dagger a_y + a_y^\dagger a_x).$$  (4)

(b) Calculate for the state found in (a): the correction in energy up to first order.

(c) Express the correction in energy up to second order.

(d) Give the lowest-order correction to the wavefunction.

NOTE: The correction term to the wavefunction is given by

$$|\psi_n^1\rangle = \sum_{m \neq n} \frac{\langle \psi_0^m | H_1 | \psi_0^n \rangle}{E_0^n - E_0^m} |\psi_0^m\rangle$$  (5)
Problem 4.
A system has unperturbed energy eigenstates $|n\rangle$ with eigenvalues $E_n$ (for $n = 0, 1, 2, 3 \ldots$) of the unperturbed Hamiltonian. It is subject to a time-dependent perturbation

$$H_I(t) = \frac{\hbar A}{\sqrt{\pi} \tau} e^{-t^2/\tau^2}$$  \hspace{1cm} (6)

where $A$ is a time-independent operator.

(a) Suppose that at time $t = -\infty$ the system is in its ground state $|0\rangle$. Show that, to first order in the perturbation, the probability that the system will be in its $m$th excited state $|m\rangle$ (with $m > 0$) at time $t = +\infty$ is:

$$P_m = a \langle m|A|0\rangle^2 e^{-b \tau (E_0 - E_m)^2}.$$  \hspace{1cm} (7)

Calculate the constants $a$, $b$, $c$ and $d$. [You may find the integral $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$ to be useful.]

(b) Next consider the limit of an impulsive perturbation, $\tau \to 0$. Find the probability $P_0$ that the system will remain in its ground state. Find a way of writing the result in terms of only the matrix elements $\langle 0|A^2|0\rangle$ and $\langle 0|A|0\rangle$.

**Hint:** the time evolution of states to first order in perturbation theory can be written as

$$|\psi(t)\rangle = \left[ e^{-i(t-t_0)H_0/\hbar} - \frac{i}{\hbar} \int_{t_0}^{t} dt' e^{-i(t-t')H_0/\hbar} H_I(t') e^{-i(t'-t_0)H_0/\hbar} \right] |\psi(t_0)\rangle$$  \hspace{1cm} (8)

where $H_0$ is the unperturbed time-independent Hamiltonian.