Problem 1:

(a) Find $\sigma_x^2\sigma_p^2$ for an eigenstate, $|n\rangle$, of a harmonic oscillator with natural frequency $\omega$.

An exact expression, not a lower bound, is desired. $\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle$ is the variance associated with a measurement of the position, and $\sigma_p^2 = \langle (p_x - \langle p_x \rangle)^2 \rangle$ is the variance associated with a measurement of the momentum.

(b) Compare your answer to that which would be found for a classical harmonic oscillator of the same energy but undetermined phase where

$$x(t) = x_0 \sin(\omega t + \phi)$$
$$p(t) = p_0 \cos(\omega t + \phi).$$
Problem 2:

Consider the “hydrogen atom problem” in two dimensions. The electron is constrained to move in a plane and feels a potential \( V(r) = -\frac{Z e^2}{r} \) due to a charge \( +Ze \) at the origin. (This mathematical model has a physically realizable analog in the physics of semiconductors.)

(a) Find the eigenfunctions and eigenvalues for the \( z \)-component of angular momentum

\[
\hat{L}_z = x\hat{p}_y - y\hat{p}_x = -i\hbar \frac{\partial}{\partial \phi}
\]

(b) The time independent Schrödinger equation for this problem is

\[
\left( -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right) \psi(r,\phi) = E\psi(r,\phi)
\]

where \( \mu \) is the reduced mass. Show that it is satisfied by a \( \psi \) which is a product of radial and angular functions: \( \psi(r,\phi) = R(r)\Phi(\phi) \). Find \( \Phi(\phi) \) and write down the equation determining \( R(r) \).

(c) What condition must be satisfied in order for \( R(r) = \alpha e^{-\frac{r}{r_0}} \) to be a solution of the radial equation? When this condition is satisfied find \( r_0 \) and the associated energy eigenvalue \( E \) in terms of \( \hbar, \mu, Z, \) and \( e \). \( (\alpha \) is a normalization constant which you need not find.)

(d) Let \( R(r) = r^{\frac{3}{2}} u(r) \). Find the equation which determines \( u(r) \). Comment on the form of this equation.

(e) A complete solution of the problem would show that the total degeneracy of the \( n^\text{th} \) bound state energy eigenvalue is \( 2n - 1 \). Draw an energy level diagram in which the levels are separated into different angular momentum “ladders”. Indicate the degeneracy and number of radial nodes associated with each of these “sub-levels” for the lowest 4 values of \( E \).
Problem 3:

A quantum mechanical particle of mass $m$ moves in one dimension in a potential consisting of two negative delta-function spikes, located at $x = \pm a$:

$$V(x) = -\lambda[\delta(x - a) + \delta(x + a)],$$

where $\lambda$ is a positive constant.

(a) Prove that the basis of bound state wave functions can be chosen so that they are each either even or odd under reflection $x \to -x$.

(b) Derive a (transcendental) equation for the binding energy of an even bound state. By sketching the functions involved, show that there is one and only one even bound state for each value of $\lambda$.

(c) Derive the transcendental equation for an odd bound state. Show that there is a minimum value of $\lambda$ for there to be an odd bound state, and determine that value.
Problem 4:

An atom that is otherwise spherically symmetric has an electron with orbital angular momentum \( \ell = 2 \) and spin \( s = 1/2 \).

(a) Using the raising and lowering operator formalism for general angular momenta, e.g.

\[
J_-|j, m\rangle = (J_x - i J_y)|j, m\rangle = \sqrt{(j + m)(j - m + 1)}\hbar|j, m - 1\rangle,
\]

construct the properly normalized linear combinations of \(|\ell, m_\ell, s, m_s\rangle\) eigenstates that have total angular momentum eigenvalues:

(i) \( j = 5/2, \ m_j = 5/2 \)

(ii) \( j = 5/2, \ m_j = 3/2 \)

(iii) \( j = 3/2, \ m_j = 3/2 \).

(b) In an external magnetic field in the \( z \) direction of magnitude \( B \), the magnetic interaction Hamiltonian is:

\[
H_{mag} = \frac{eB}{2mc}(L_z + 2S_z).
\]

What energy corrections are induced for the states of part (a), for a weak field \( B \)?