Quantum Mechanics

You may solve ALL four problems! The **three best graded** count towards your total score.

(40 points each; total possible score: 120 points)

I. **RECTANGULAR POTENTIAL BARRIER** [(5+20+10+5) PTS]

A particle with energy \(E\) encounters a rectangular potential barrier in one dimension, \(V(x) = V_0\) for \(0 \leq x \leq a\) and \(V(x) = 0\) for \(x < 0\) and \(x > a\), coming from the left. Here \(V_0 > 0\) and \(a > 0\).

a) Write down the Schrödinger equation, make an Ansatz for the wave function including the wave numbers, and give the boundary conditions at \(x = 0\) and \(x = a\).

b) Calculate the transmission coefficient, \(t\), as a function of the two wave numbers and \(a\).

c) The related transmission probability \(T = |t|^2\) is

\[
T = \left(1 + \frac{\sin^2(k_{I\!I}a)}{4\epsilon(\epsilon - 1)}\right)^{-1},
\]

with \(\epsilon = E/V_0\) for \(\epsilon > 1\) and \(k_{I\!I}\) the wave numbers for \(0 < x < a\). What is the lowest energy at which the transmission probability becomes 1? What would this energy be for a classical particle? What is the tunneling probability for \(E < V_0\) (the same expression as above holds, but write \(T\) as a function of real quantities only!)?

d) Sketch \(T\) as function of \(\epsilon\) such that it shows the correct physical behavior for \(\epsilon < 1\), \(\epsilon = 1\), and \(\epsilon > 1\).

*Hint for b*: The boundary conditions give you four equations for the six unknown coefficients of the wave functions. However, the definition of the problem (particle comes from the left!) defines two of the coefficients (explain in your answer)! Then you only need to solve the linear equation system for the transmission coefficient.

II. **PARTICLE IN A BOX** [(6+4+30) PTS]

A spinless particle of charge \(e\) and mass \(m\) is confined to a cubic box of side \(L\). A weak uniform electric field \(E_0\) is applied, with direction parallel to one of the sides of the box, and the electrostatic potential for the field is taken to be zero at the center of the cubic box.

a) Write down the unperturbed energy eigenvalues and corresponding normalized wave functions.

b) Show explicitly that, to first order in the perturbation \(E_0\), the ground state energy is unchanged.

c) Find the change in the ground state energy at second order in \(E_0\). You should leave your answer in terms of an infinite sum of the form

\[
\sum_{n=2,4,6,...} \frac{n^p}{(n^2 - 1)^q}
\]

where \(p\) and \(q\) are certain integers that you will find. How does the change in the ground state energy scale with the size of the box \(L\)?

*Hint: you may find the following definite integrals useful:

\[
\int_{-L/2}^{L/2} du \sin\left(\frac{n\pi u}{L} + \frac{n\pi}{2}\right)\sin\left(\frac{n'\pi u}{L} + \frac{n'\pi}{2}\right) = \begin{cases} 
\frac{L}{2} & (n = n') \\
0 & (n \neq n')
\end{cases}
\]
and
\[ \int_{-L/2}^{L/2} du \sin \left( \frac{n\pi u}{L} + \frac{n\pi}{2} \right) \sin \left( \frac{n'\pi u}{L} + \frac{n'\pi}{2} \right) = \begin{cases} \frac{4L^2}{\pi^2} \frac{nn'}{(n^2 - n'^2)^2} & (n + n' = \text{odd}) \\ 0 & (n + n' = \text{even}) \end{cases} \]

III. SPIN [(20+20) PTS]

A particle of spin 1/2 is described by the eigenspinor
\[ \frac{1}{\sqrt{5}} \begin{pmatrix} i \\ 2 \end{pmatrix} \]

a) What are the values of \( \langle S_z \rangle \), \( \langle S_y \rangle \), \( \langle S_y^2 \rangle \), and \( \Delta(S_y) = \langle (S_y - \langle S_y \rangle)^2 \rangle \)?

b) What is the probability that a measurement of \( S_y \) gives the value \( \hbar/2 \)?

Hint: The Pauli matrices are on the formula sheet.

IV. TWO SPRINGS [(10+10+10+10) PTS]

Let us consider the Hamiltonians
\[ H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad \text{and} \quad H_1 = \frac{1}{2} m\omega_1^2 x^2 \]

a) Determine the eigenvalues of the Hamiltonian \( H_{\text{tot}} = H + H_1 \).

b) Let us take the solutions of \( H \) to be \( |n\rangle \) (note that these are not the solutions of \( H_1 \) or \( H_{\text{tot}} \)). Calculate the matrix elements \( \langle n' | H_1 | n \rangle \). Show that for the following matrix elements we can write \( \langle 0 | H_1 | 0 \rangle = \frac{1}{2} \alpha \), \( \langle 2 | H_1 | 2 \rangle = \frac{5}{2} \alpha \), and \( \langle 0 | H_1 | 2 \rangle = \langle 2 | H_1 | 0 \rangle = \alpha/\sqrt{2} \). Determine \( \alpha \).

c) Let us now consider the situation where \( \omega_1 \ll \omega \). In this limit we can take as a trial solution for the change in the lowest eigenstate of \( H \) due to the presence of \( H_1 \),
\[ |\psi\rangle = \cos \theta |0\rangle + \sin \theta |2\rangle, \]

where \( \cos \theta \) and \( \sin \theta \) are coefficients. Explain why this is a reasonable trial wave function and determine the \( \theta \) value that minimizes \( \langle \psi | H_{\text{tot}} | \psi \rangle \).

d) Show that the exact eigenfunction for \( H_{\text{tot}} = H + H_1 \) with \( n_{\text{tot}} = 0 \), is indeed a combination of the \( n = 0 \) and \( n = 2 \) eigenfunctions of \( H \) in the limit \( \omega_1 \ll \omega \). (There is no need to find the exact combinations, only show that it contains a combination of the right Hermite polynomials).

The solutions of \( H \) for \( n = 0 \) and 2 are
\[ \varphi_0(x) = \frac{1}{\sqrt{2}} \left( \frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\pi \hbar} x^2} \]
\[ \varphi_2(x) = \frac{1}{2\sqrt{2}} \left( \frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\pi \hbar} x^2} \left( \frac{4m\omega}{\hbar} x^2 - 2 \right) \]
NIU Physics PhD Candidacy Exam - Spring 2017

Quantum Mechanics

You may solve ALL FOUR PROBLEMS! The three best graded count towards your total score.

(40 points each; total possible score: 120 points)

I. SPHERICAL POTENTIAL WELL [(10+10+10+10) PTS]

Consider a spherical potential well with

\[ U(r) = \begin{cases} 0 & r < a \\ \infty & r > a \end{cases} \]

a) Give the radial part of the Hamiltonian for \( r < a \).

b) We are now only interested in the wavefunction for \( l = 0 \). Give the general solution of the differential equation by trying a solution \( R = P/r \). (Normalization is not necessary).

c) Determine the behavior of the wavefunction for \( r \to 0 \). What does the condition that the wavefunction should not diverge at \( r = 0 \) imply?

d) Determine the eigenvalues.

II. SPINS IN THE BOX [(22+9+9) PTS]

Two identical spin-1/2 fermions of mass \( M \) are confined in a cubic box of side \( L \). The sides of the box have infinite potential. The identical fermions interact according to the attractive potential:

\[ V(\vec{r}_1, \vec{r}_2) = -\epsilon L^3 \delta^3(\vec{r}_1 - \vec{r}_2), \]

where \( \epsilon \) is small and positive, and should be treated as a perturbation. Do not neglect the effects of spin degrees of freedom in this problem. [Hint: you may or may not find \( \cos \theta = (e^{i\theta} + e^{-i\theta})/2 \) and/or \( \sin \theta = (e^{i\theta} - e^{-i\theta})/2i \) to be useful in doing an integral.]

a) Find the total spin, the degeneracy, and the energy of the ground state, working to first order in \( \epsilon \).

b) What is the total spin and the degeneracy of the first excited state?

c) If the potential is instead repulsive (\( \epsilon < 0 \)), what is the total spin and the degeneracy of the first excited state?

III. PERTURBED BOUND STATES [(5+5+12+18) PTS]

Consider a particle of mass \( m \) in a two-dimensional infinite square well of width \( a \)

\[ V_0(x, y) = \begin{cases} 0 & 0 \leq x \leq a, \ 0 \leq y \leq a \\ \infty & \text{otherwise} \end{cases} \]

a) Write down the time-independent Schrödinger equation for this problem.

b) Write down the energy eigenfunctions and corresponding energy eigenvalues for the ground and first excited states.
   (You need not derive the answer if you know it.)

We now add a time-independent perturbation

\[ V_1(x, y) = \begin{cases} \lambda xy & 0 \leq x \leq a, \ 0 \leq y \leq a \\ 0 & \text{otherwise} \end{cases} \]

c) Obtain the first-order energy shift for the ground state.

d) Obtain the zeroth-order energy eigenfunctions and the first-order energy shifts for the first excited states.
IV. CHARGED 1D LINEAR OSCILLATOR [(20+10+10) PTS]

Let us consider an one-dimensional harmonic oscillator in a uniform electric field applied along the oscillation axis, i.e., with additional potential \( U(\hat{x}) = -eE\hat{x} \).

a) Find the change in energy levels of the charged linear oscillator.

b) Find the change in the eigenfunctions of the charged linear oscillator. Express the wavefunctions for \( E \neq 0 \) in terms of the eigenfunctions of the one-dimensional harmonic oscillator.

c) Find the polarizability of the oscillator in these eigenstates. The polarizability, \( \alpha \), determines the mean dipole moment, \( p = \alpha E \), induced by the weak external electric field in an isotropic system. It can also be expressed as the negative second derivative of the energy with respect to the electric field. How does \( \alpha \) depend on the quantum number?
I. ATOMIC ORBITALS \((10+10+10+10) \text{PTS}\)

We want to study the change in energy of atomic orbitals feeling a potential energy \(\hat{H}_1\) due to a constant electric field in the \(x\) direction, where \(\hat{H}_1\) is given by

\[
\hat{H}_1 = eE_x x.
\]

We will consider the coupling of the \(n, l = 0\) orbital to \(n', l = 1\) orbitals, where the unperturbed orbitals are given by

\[
\psi_{n00}(r, \theta, \varphi) = R_{n0}(r)Y_{00}(\theta, \varphi) \quad \text{and} \quad \psi_{n'1m}(r, \theta, \varphi) = R_{n'1}(r)Y_{1m}(\theta, \varphi)
\]

a) Express \(x\) in terms of \(r\) and spherical harmonics.

b) Evaluate the matrix element \(\langle n'1m | \hat{H}_1 | n00 \rangle\). Express the radial part of this matrix element in terms of integrals involving \(R_{n0}\) and \(R_{n'1}\). For the angular part, make use of the fact that \(Y_{00}\) is a constant and the normalization condition of the spherical harmonics. Show that the result can be written as

\[
\langle n'1m | \hat{H}_1 | n00 \rangle = P(\delta_{m,-1} - \delta_{m,1}),
\]

where \(P\) is the factor containing the radial parts of the matrix element of \(H_1\).

c) Using the result from b), set up the \(4 \times 4\) Hamiltonian in matrix form for the basis set \(|n00\rangle, |n'11\rangle, |n'1, -1\rangle, |n'10\rangle\). Include also an energy difference between the states with different \(n\), i.e. \(\Delta = \langle n'1m | \hat{H} | n'1m \rangle - \langle n00 | \hat{H} | n00 \rangle\), where \(\hat{H}\) is the unperturbed Hamiltonian of the hydrogen atom.

d) Find the eigenenergies for the Hamiltonian obtained in c).

II. POSITRONIUM \((20+20) \text{PTS}\)

In the ground orbital state of positronium (an electron-positron bound state), the Hamiltonian in the presence of an external magnetic field \(B \hat{z}\) is given to a good approximation by

\[
\hat{H} = a\hat{S}_e \cdot \hat{S}_p + b(S_z^e - S_z^p)
\]

where \(a\) is a constant and \(b\) is proportional to \(B\), and \(\hat{S}_e\) and \(\hat{S}_p\) are the spin operators for the electron and positron respectively.

a) Find the energy eigenvalues and eigenstates of \(\hat{H}\).

b) Now suppose the magnetic field is off, so \(b = 0\). At time \(t = 0\), the electron and the positron spin components in the \(z\) direction are measured to be up and down, respectively. What is the probability that the electron spin will be down when measured again at a later time \(t\)?

III. ATTRACTIVE POTENTIAL \((30+10) \text{PTS}\)

Let us consider a particle with mass \(m\) in a one-dimensional square potential well with an attractive delta-potential at its center, i.e.,

\[
U(x) = \begin{cases} 
\infty & x \leq -a \\
-U_0\delta(x) & -a < x < a \\
\infty & x \geq a
\end{cases}
\]
where $2a$ is the width of the square well and $U_0$ the strength of the delta potential.

a) Derive the equation determining the eigenenergy $\epsilon$ of the bound eigenstate of the delta potential.

*Hint:* This equation has the form $\tanh(ka) = \alpha k U_0$ with a constant $\alpha$ you should find and wave number $\hbar k = \sqrt{2m|\epsilon|}$.

b) What is the minimum $U_0$ for which a state with $\epsilon < 0$ exists?

IV. 1D HARMONIC OSCILLATOR [(10+10+20) PTS]

Given is a 1D harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

and a wave function which is a mixture of the $n = 0$ and $n = 1$ states

$$\psi(x) = \frac{1}{\sqrt{5}}(u_0(x) - 2u_1(x)).$$

a) Draw $\psi(x)$

b) What is $\langle E \rangle$ in terms of $m$ and $\omega$?

c) What are $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, and $\delta x$?
NIU Physics PhD Candidacy Exam - Spring 2016

Quantum Mechanics

Do Only Three Out Of Four Problems

I. INTERACTING SPINS [(16+8+16) PTS]

We consider two spin-1/2 particles interacting via the operator

\[ f = a + b \mathbf{s}_1 \cdot \mathbf{s}_2 \]

where \( a \) and \( b \) are constants and \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) are the spin operators for particles 1 and 2. The total spin angular momentum is \( \mathbf{j} = \mathbf{s}_1 + \mathbf{s}_2 \).

a) Show that \( f \), \( \mathbf{j}^2 \) and \( j_z \) can be measured simultaneously.

b) Derive the matrix representation for \( f \) in the \( |j, m, s_1, s_2\rangle \) basis. (Label rows and columns of your matrix.)

c) Derive the matrix representation for \( f \) in the \( |s_1, s_2, m_1, m_2\rangle \) basis. (Again, label rows and columns of your matrix.)

II. POTENTIAL WELL [(14+13+13) PTS]

Let the potential \( V = 0 \) for \( r < a_0 \) (the Bohr radius) and \( V = \infty \) for \( r > a_0 \). \( V \) is a function of \( r \) only.

a) What is the energy of an electron in the lowest energy state of this potential?

b) How does that compare to the energy of the 1S state of Hydrogen?

c) What is the approximate energy of the lowest energy state with angular momentum greater than 0? Do not try to solve the equation but instead base your approximation on the shape of the well.

III. HARMONIC OSCILLATOR IN ELECTRIC FIELD [(6+14+20) PTS]

Consider a simple harmonic oscillator in one dimension with the usual Hamiltonian

\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{m \omega^2}{2} \hat{x}^2. \]

The eigenfunction of the ground state can be written as \( \psi_0(x) = N e^{-\alpha^2 x^2/2} \).

a) Determine the constant \( N \).

b) Calculate both the constant \( \alpha \) and the energy eigenvalue of the ground state.

c) At time \( t = 0 \), an electric field \( |E| \) is switched on, adding a perturbation of the form \( \hat{H}_1 = e|E|\hat{x} \) to the Hamiltonian. What is the new ground state energy?

IV. EXPANDING POTENTIAL WELL [(12+16+12) PTS]

Let us take an infinite potential well in one dimension of width \( L \) with a particle of mass \( m \) moving inside it \( (-L/2 < x < L/2) \). The particle is initially in the lowest eigenstate.

a) Calculate the energy and wavefunction.
b) At $t = 0$, the walls of the potential well are suddenly moved to $-L$ and $L$. Calculate the probability of finding the particle in the eigenstates of the new system.

c) What is the expectation value of the energy in the new eigenstates?
(You can make use of the series $\sum_{m=0}^{\infty} \frac{(2m+1)^2}{(2m+1)^2 - 4} = \frac{\pi^2}{16}$).
I. POTENTIAL WELL \((8+10+10+12)\) PTS

Let us consider a particle of mass \(m\) in an infinite square potential well \(V(x) = 0\) for \(|x| < x_0\), \(\infty\) otherwise. The potential well is depicted in the figure.

a) Redraw the potential well and sketch the lowest three eigenfunctions (for \(n = 1, 2, 3\) with \(n\) being the quantum number).

b) Calculate the eigenenergies \(E_n\) of the particle. (Hint: \(E_n = \kappa n^2 \hbar^2 / 2m x_0^2\) with a numerical prefactor \(\kappa\), which you need to calculate.)

Next, we consider the wave function

\[
\psi(x) = \gamma \sqrt{x_0} \left[ \cos \left( \frac{\pi x}{2x_0} \right) + 2 \sin \left( \frac{\pi x}{x_0} \right) \right],
\]

which is a linear combination of the lowest two eigenfunctions.

c) Find the value of \(\gamma\), such that \(\psi(x)\) is normalized.

d) Calculate the expectation value of the kinetic energy.

II. HARMONIC OSCILLATOR \((20+20)\) PTS

Given is a 2D harmonic oscillator with Hamiltonian

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m \omega^2}{2} (\hat{x}^2 + \hat{y}^2) + km \hat{x} \hat{y},
\]

with \(\hat{p} = (\hat{p}_x, \hat{p}_y)\).

a) For \(k = 0\), what are the energies of the ground state and first and second excited states? What are the degeneracies of each state?

b) For \(k > 0\), using first order perturbation theory, what are the energy shifts of the ground state and the first excited states?

III. SPIN IN A MAGNETIC FIELD \((20+20)\) PTS

The Hamiltonian for a spin \(s\) of a particle with charge \(e\) in an applied magnetic field \(B\) is given by

\[
\hat{H} = -\frac{ge}{2m} \hat{s} \cdot B,
\]

where \(g\) is the gyromagnetic ratio.

a) Calculate \(d\hat{s}/dt\).

b) Describe the motion if the magnetic field is in the \(y\) direction. Express the results in terms of the initial spin components.
IV. TWO FERMIONS \((10+10+20) \text{PTS}\)

Suppose that two identical spin-1/2 fermions, each of mass \(m\), interact only via the potential

\[
V(r) = \frac{4}{3} \frac{a^2 \vec{S}_1 \cdot \vec{S}_2}{r}
\]

where \(r\) is the distance between the particles, and \(\vec{S}_1\) and \(\vec{S}_2\) are the spins of particles 1 and 2 respectively, and \(a\) is a constant.

a) What is the value of the total spin for the bound states of the system?

b) What values of the orbital angular momentum are allowed for bound states?

c) Find the energies and degeneracies of the ground state and the first two excited states of the system.
I. FLUX QUANTUM [(5+10+20+5) PTS]

A charged particle (charge $-e$) moves in a (three dimensional) space having an infinite, impenetrable cylinder of radius $a$ along the z-axis in its center. $\psi_0$ shall be the solution of the stationary Schrödinger equation outside the cylinder without magnetic field.

a) Now we apply a magnetic field $B$ to the system, which is determined by the vector potential, $A$. Write down the corresponding Schrödinger equation.

b) Here the vector potential will be

$$A(r, \varphi, z) = \begin{cases} \frac{1}{2} Br\hat{\varphi}, & r < a \\ \frac{a^2 B}{2r}\hat{\varphi}, & r > a \end{cases}$$

where $\hat{\varphi}$ is the angular unit vector. Calculate the magnetic field distribution, $B$, from this vector potential.

c) Use the functional form $\psi = e^{-i\gamma \chi} \psi_0$ with

$$\chi(x) = \int_{x_0}^{x} A \cdot ds$$

for the wave function and solve the Schrödinger equation corresponding to the above vector potential. Find the constant $\gamma$.

d) For which flux $Ba^2$ is the wave function $\psi$ unique?

**Hint:** $\nabla = \left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z} \right)$, $\nabla \times \mathbf{v} = \frac{1}{r} \begin{vmatrix} \hat{r} & \hat{\varphi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ v_r & v_\varphi & v_z \end{vmatrix}$.

II. NEUTRONS IN THE BOX [(10+10+10+10) PTS]

Eight non-interacting neutrons are confined to a 3D square well of size $D = 5\text{fm}$ ($5 \cdot 10^{-15}\text{m}$) such that $V = -50\text{MeV}$ for $0 < x < D$, $0 < y < D$, $0 < z < D$ and $V = 0$ everywhere else.

a) How many energy levels are there in this well?

b) What is the degeneracy of each energy level?

c) What is the approximate Fermi energy for this system?

d) What is the relative probability to be in the lowest energy state to the fourth lowest energy state at $k_B T = 10\text{MeV}$? Just write down the ratio (do not calculate the value).

**Useful constants:** mass of neutron = 940 MeV/$c^2$, $ch = 197$ MeV·fm, $hc = 1240$ MeV·fm
III. ATTRACTIVE WALL [(12+12+8+8) PTS]

Let us consider a step potential with an attractive $\delta$-function potential at the edge

$$U(x) = U\theta(x) - \frac{\hbar^2 g}{2m} \delta(x).$$

a) Calculate the wave function for $E > V$.

b) Calculate the reflection coefficient $|R|^2$ and discuss the limit $E \gg U \gg \hbar^2 g/2m$.

c) Determine the wavefunction for the bound state.

d) What is the energy of the bound state?

IV. $K$-CAPTURE [(13+13+14) PTS]

The $K$-capture process involves the absorption of an inner orbital electron by the nucleus, resulting in the reduction of the nuclear charge $Z$ by one unit. This process is due to the non-zero probability that an electron can be found within the volume of the nucleus. Suppose that an electron is in the 1$s$ state of a Hydrogen-like potential, with wavefunction given by:

$$\psi(r) = \frac{1}{\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}$$

where $a_0$ is the Bohr radius.

a) Calculate the probability that a 1$s$ electron will be found within the nucleus. Take the nuclear radius to be $R$, which you may assume is much smaller than $a_0$.

b) Assume that an electron is initially in the ground state with $Z = 2$, and a nuclear reaction abruptly changes the nuclear charge to $Z = 1$. What is the probability that the electron will be found in the ground state of the new potential after the change in nuclear charge?

c) Now assume instead that the nuclear reaction leaves the electron in a state given by the wavefunction

$$\Psi(r, \theta, \phi) = A(\sin \theta \sin \phi + \sin \theta \cos \phi + \cos \theta) r e^{-r/a_0},$$

where $A$ is an appropriate normalization constant. What are the possible values that can be obtained in measurements of $L^2$ and $L_z$, and with what probabilities will these values be measured?
I. ELECTRON IN WELL [(12+10+18) PTS]

An electron (ignore the effects of spin) is in a 1-dimensional potential well $V(x)$ with

$$V(x) = \begin{cases} 
0 \text{ eV} & \text{for } 0 < x < 0.4 \text{nm and } x > 10.4 \text{nm} \\
200 \text{ eV} & \text{for } x < 0 \text{ and } 0.4 \text{nm} < x < 10.4 \text{nm}
\end{cases}$$

a) What are the lowest two energies (in eV) for the bound states in the region with $0 < x < 0.4 \text{nm}$? How do they compare to the energies levels of an infinite well with the same width?

b) How many bound states are in this well?

c) If an electron is initially in the well, what is the relative probability for an electron in the lowest energy state to tunnel through the barrier compared to an electron in the second lowest energy state? Give the result in a number good to a factor of 10.

Electron mass $= 0.5 \text{MeV/c}^2$; $\hbar c = 197 \text{ eV nm}$; $\hbar c = 1240 \text{ eV nm}$

II. SLIGHTLY RELATIVISTIC 1D HARMONIC OSCILLATOR [(6+10+12+12) PTS]

You know that the concept of potential energy is not applicable in relativistic situations. One consequence of this is that the only fully relativistic quantum theories possible are quantum field theories. However, there do exist situations where a particle’s motion is “slightly relativistic” (e.g., $v/c \sim 0.1$) and where the force responds quickly enough to the particle’s position that the potential energy concept has approximate validity.

Here we consider the one-dimensional harmonic oscillator, defined by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + m\omega^2 \hat{x}^2.$$ 

Reminder: The eigenvalues of the harmonic oscillator are $E_n = \hbar \omega \left(n + \frac{1}{2}\right)$ and the eigenstates can be expressed as $|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle$ with the creation ($\hat{a}^\dagger$)/annihilation ($\hat{a}$) operators given by $(\hat{x}/x_0 \pm i\hat{p}/p_0)/\sqrt{2}$, respectively $[x_0 = \sqrt{\hbar/(m\omega)}$, $p_0 = \sqrt{m\hbar\omega}]$, $\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ and $\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$.

a) Using the relativistic energy relation $E = (m^2 c^4 + p^2 c^2)^{1/2}$, derive the $\hat{p}^4$ order correction to the harmonic oscillator Hamiltonian given above.

b) Calculate $\langle n | \hat{p}^4 | 0 \rangle$.

Hint: The result has the form $p_0^4 (C_0 \delta_{n,0} + C_2 \delta_{n,2} + C_4 \delta_{n,4})$, with numerical prefactors $C_i$ that you will find.

c) Calculate the leading non-vanishing energy shift of the ground state due to this relativistic perturbation.

d) Calculate the leading corrections to the ground state eigenvector $|0\rangle$. 
III. COUPLED SPINS \( [18+6+16] \) PTS

Consider two spatially localized spins 1/2, \( S_1 \) and \( S_2 \), coupled by a transverse exchange interaction and in an inhomogeneous magnetic field. The Hamiltonian is:

\[
H = b_1 S_{1z} + b_2 S_{2z} - k (S_1^+ S_2^- + S_1^- S_2^+),
\]

where \( b_1 \) and \( b_2 \) are proportional to the magnetic fields at the two sites and \( k \) measures the strength of the exchange coupling, and \( S_1^\pm = S_1x \pm iS_1y \) and \( S_2^\pm = S_2x \pm iS_2y \).

a) Find the energy eigenvalues.

b) If both spins are in the \(+z\) directions at time \( t = 0 \), what is the probability that they will both be in the \(+z\) direction at a later time \( t \)?

c) If \( b_1 = b_2 \), and at time \( t = 0 \) the spin \( S_1 \) is in the \(+z\) direction and \( S_2 \) is in the \(-z\) direction, then what is the probability that \( S_1 \) will be in the \(-z\) direction at a later time \( t \)?

IV. SPHERICAL POTENTIAL WELL \( [10+10+10+10] \) PTS

Consider a spherical potential well with

\[
U(r) = 0 \quad r < a \quad ; \quad U(r) = \infty \quad r > a
\]

a) Give the radial part of the Hamiltonian for \( r < a \).

b) We are now only interested in the wavefunction for \( l = 0 \). Give the general solution of the differential equation by trying a solution \( R = P/r \). (Normalization is not necessary).

c) Determine the behavior of the wavefunction for \( r \to 0 \). What does the condition that the wavefunction should not diverge at \( r = 0 \) imply?

d) Determine the energy eigenvalues.
I. POTENTIAL STEP [(10+15+15) PTS]

Let us consider a particle in a potential \( V(x) = V_0 \Theta(-x) \). As shown in the figure, the particle shall come from the left with energy \( E > V_0 \) towards the step at \( x = 0 \).

a) Make an ansatz for the wave function in regions \( x < 0 \) (I) and \( x > 0 \) (II) and solve the Schrödinger equation in these regions.

b) Use the continuity conditions at \( x = 0 \) and determine the coefficients you used in a) as functions of \( V_0 \) and \( E \).

c) Calculate the probability that the particle is reflected at the potential step.

II. BASIS TRANSFORMATION [(10+14+16) PTS]

Let us consider a three-dimensional basis \( |0\rangle, |k_0\rangle, \) and \( |−k_0\rangle \) where the basis functions are given by

\[
\psi_k(x) = (x|k) = \frac{1}{\sqrt{2\pi}} e^{ikx}.
\]

a) Write the Hamiltonian

\[
H = -2|0\rangle \langle 0 | - \sum_{k,k'=0,\pm k_0; k \neq k'} |k\rangle \langle k'| 
\]

in matrix form.

b) We want to transform the basis from exponential into sines and cosines. We can do this by combining the \( |\pm k_0\rangle \) using the unitary transformation

\[
|\psi\rangle = U|\psi'\rangle \quad \text{with} \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.
\]

Express the Hamiltonian in the new basis.

c) Calculate the eigenvalues and eigenvectors of the Hamiltonian (Express the result in bra-ket notation).

III. BOUND STATES [(14+10+16) PTS]

Let the potential \( V = 0 \) for \( r < a_0 \) (the Bohr radius) and \( V = \infty \) for \( r > a_0 \). \( V \) is a function of the radius \( r \) only.

a) What is the energy of an electron in the Bohr radius and \( V = \infty \) for \( r > a_0 \). \( V \) is a function of the radius \( r \) only.

b) How does that compare to the energy of the 1S state of Hydrogen?

c) What is the approximate energy of the lowest energy state with angular momentum greater than 0 (you can leave this result in integral form)?
IV. SPIN PRECESSION [(8+8+8+8+8) PTS]

We consider a spin-1/2 particle in an external magnetic field $B_z$, i.e., the Hamiltonian is $H = -\mu B_z S_z$ with spin magnetic moment $\mu$ and $S_z$ is the $z$ component of the spin operator.

a) Suppose the particle is initially (time $t = 0$) in an eigenstate $|+, x\rangle$ of spin $S_x$ with eigenvalue $+\hbar/2$. Express $|+, x\rangle$ in terms of the eigenstates $|+, z\rangle$ and $|-, z\rangle$ of spin $S_z$.

b) Evaluate the time evolution of the state $|+, x\rangle$ in the Schrödinger picture.

c) Evaluate the time evolution of the expectation value $\langle S_x \rangle$ in the Schrödinger picture, assuming that the particle is at time $t = 0$ in the state $|+, x\rangle$.

d) Solve the Heisenberg equation of motion for the operator $S_H^x(t)$ in the Heisenberg picture.

e) Show that the time-dependent expectation value $\langle S_H^x(t) \rangle$ in the Heisenberg picture equals your result from (c) obtained in the Schrödinger picture.
I. PARTICLE IN POTENTIAL WELL \([\{16+12+12\} \text{ PTS}\]

Here we consider a particle of mass \(m\) confined in a one-dimensional potential well defined by
\[
U(x) = \begin{cases} \alpha \delta(x), & |x| < a \\ \infty, & |x| \geq a \end{cases}
\]
for \(a > 0\) and \(\alpha > 0\). The energy levels (eigenvalues) \(E_n\) can be calculated without perturbation theory.

a) For \(m a a \hbar^2 \gg 1\), show that the lowest energy levels \(n \sim 1\) are pairs of close lying levels.

b) Find the spectrum for large energies \(n \gg 1\).

c) Find the energy levels for \(a < 0\).

II. DENSITY MATRIX \([\{10+10+5+15\} \text{ PTS}\]

Let us consider a system in a (normalized) pure quantum state \(|\psi\rangle\) and define the operator
\[
\hat{\rho} = |\psi\rangle \langle \psi|,
\]
which is called the density matrix.

a) Show that the expectation value of an observable associated with the operator \(\hat{A}\) in \(|\psi\rangle\) is \(\text{Tr}(\hat{\rho}\hat{A})\).

b) Frequently physicists don’t know exactly which quantum state their system is in. (For example, silver atoms coming out of an oven are in states of definite \(\mu\) projection, but there is no way to know which state any given atom is in.) In this case there are two different sources of measurement uncertainty: first, we don’t know what state they system is in (statistical uncertainty, due to our ignorance) and second, even if we did know, we couldn’t predict the result of every measurement (quantum uncertainty, due to the way the world works). The density matrix formalism neatly handles both kinds of uncertainty at once. If the system could be in any of the states \(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_i\rangle, \ldots\) (which do not necessarily form a basis set), and if it has probability \(p_i\) of being in state \(|\psi_i\rangle\), then the density matrix
\[
\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i|,
\]

is associated with the system. Show that the expectation value of the observable associated with \(\hat{A}\) is still given by \(\text{Tr}(\hat{\rho}\hat{A})\).

c) Calculate \(\text{Tr}(\hat{\rho})\).

d) Now we consider the example of two spin-\(1/2\) particles. Give the density matrix for the following cases in \(\{|+\rangle, \{-\rangle\}\)-basis (\pm refer to the sign of the eigenvalue of the spin operator):

- Both spins have the same orientation.
- The total spin is zero.

Hint: The trace can be written as \(\text{Tr}(\hat{A}) = \sum_k \langle \phi_k | \hat{A} | \phi_k\rangle\), where the \(|\phi_k\rangle\) form an orthonormal basis.
III. ELECTRON IN A HOMOGENEOUS MAGNETIC FIELD [(16+12+12) PTS]

Let us consider electrons (mass $\mu$, charge $-e$), which move in a spatially homogeneous constant magnetic field in the $x-y$ plane.

a) Show that the Hamiltonian can be written as

$$\hat{H} = \frac{2}{\mu} \left[ -\hbar^2 \frac{\partial}{\partial \eta} \frac{\partial}{\partial \bar{\eta}} - \hbar \frac{eB}{4c} \left( \frac{\partial}{\partial \eta} - \bar{\eta} \frac{\partial}{\partial \bar{\eta}} \right) + \frac{eB}{4c} \right] \eta \bar{\eta}^2,$$

using the coordinate transformation $\eta = x + iy$, $\bar{\eta} = x - iy$.

b) Use the product ansatz $\psi(\eta, \bar{\eta}) = f(\eta, \bar{\eta}) \exp(-\eta \bar{\eta}^2/(4l^2))$ with $l = \sqrt{\hbar c/(eB)}$ and construct the stationary Schrödinger equation for $f$. Show that the wave function related to the lowest eigenvalue $E_0 = \hbar \omega_c (\omega_c = eB/(mc))$ satisfies the condition $\partial_\eta f = 0$ and is therefore a polynomial in $\eta$.

c) Calculate the component of the angular momentum operator $\hat{l}_z$ perpendicular to the $x-y$ plane in complex coordinates and show that $[\hat{H}, \hat{l}_z] = 0$ and that the functions $\varphi_m(\eta) = \eta^m \exp(-\eta \bar{\eta}^2/(4l^2))$ are eigenfunctions of $\hat{l}_z$. Calculate their eigenvalues.

Hint: The momentum operator for a charged particle (charge $q$) in a magnetic field transforms as $\hat{p} \rightarrow \hat{p} - \frac{e}{c} \hat{A}$ with the vector potential for a homogeneous field being $\hat{A} = -\frac{1}{2} (r \times \hat{B})$. In order to rewrite the partial derivatives in complex coordinates, use the chain rule, e.g., express $\frac{\partial}{\partial x} f(\eta, \bar{\eta})$ in terms of $\frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial \bar{\eta}}$.

IV. ANHARMONIC OSCILLATOR [(12+12+16) PTS]

The Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V} \equiv \left[ \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \right] + \left[ \gamma \hat{x}^4 \right]$$

describes a one-dimensional anharmonic oscillator, where $\hat{V}$ is a perturbation of the harmonic oscillator $\hat{H}_0$. As we know the eigenvalues of the harmonic oscillator, $\hat{H}_0$, are $E_n^{(0)} = n \omega (n = 0, 1, 2, \ldots)$ and the eigenstates can be expressed as $|\psi_n^{(0)}\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |\psi_0^{(0)}\rangle$ with the creation (or raising) operator $\hat{a}^\dagger = (\hat{x}/x_0 + ix_0\hat{p}/\hbar)/\sqrt{2}$ [the corresponding annihilation (or lowering) operator $\hat{a} = (\hat{x}/x_0 - ix_0\hat{p}/\hbar)/\sqrt{2}$ for $n = 0, 1, 2, \ldots$ ] $x_0 = \sqrt{\hbar/(m\omega)}$. $\hat{a}^\dagger |\psi_0^{(0)}\rangle = \sqrt{n+1} |\psi_{n+1}^{(0)}\rangle$ and $\hat{a} |\psi_n^{(0)}\rangle = \sqrt{n} |\psi_{n-1}^{(0)}\rangle$.

a) Assuming that $\gamma$ is small, calculate the ground state energy, $E_0^{(1)}$, in first-order perturbation theory.

b) Calculate the ground state eigenstate, $|\psi_0^{(1)}\rangle$, in first-order perturbation theory.

c) Calculate the energy eigenvalues in first order perturbation theory for arbitrary $n$, i.e., $E_n^{(1)}$. 
Problem 1. A particle of mass $m$ and momentum $p$ is incident from the left on a one-dimensional potential well $V(x)$, which is non-zero only between $x = 0$ and $x = a$ as shown in the figure. The energy $E = p^2/2m$ of the incident particle is very large compared to the depth of the potential, so that you may treat the potential as small, and keep only effects that are leading order in $V(x)$ (the Born approximation).

(a) What is $\phi(x)$, the unperturbed wavefunction (for $V(x) = 0$)? [3 points]

(b) Let us write the perturbed wavefunction as:

$$\psi(x) = \phi(x) + \int_{-\infty}^{\infty} G(x, x') V(x') \phi(x') dx' + \ldots.$$ 

Show that $G(x, x')$ then obeys the differential equation

$$\frac{\partial^2}{\partial x^2} G(x, x') + C_1 G(x, x') = C_2 \delta(x - x')$$

where $C_1$ and $C_2$ are positive constants that you will find. [12 points]

(c) Try a solution for $G(x, x')$ of the form:

$$G(x, x') = \begin{cases} 
A e^{ik(x-x')} & \text{for } x \geq x', \\
A e^{-ik(x-x')} & \text{for } x \leq x'.
\end{cases}$$

Solve for the constants $A$ and $k$ in terms of $m$ and $p$. [12 points]

(d) Find the probability that the particle will be reflected from the well. Leave your answer in terms of a well-defined integral involving the potential $V(x)$. [13 points]
Problem 2. Consider the Hamiltonian

\[ H = \frac{p^2}{2m} - \alpha \delta(x). \]  \hspace{1cm} (1)

Although this problem can be solved exactly, let us approach it variationally and take as a guess for our ground state a Gaussian

\[ \psi(x) = Ae^{-bx^2}. \]  \hspace{1cm} (2)

(a) Find the normalization constant A. [8 points]
(b) Calculate the kinetic energy. [10 points]
(c) Calculate the potential energy (the delta function). [10 points]
(d) Find b using the variational principle. [12 points]

Problem 3. Consider the three-dimensional infinite cubical well

\[ V(x, y, z) = \begin{cases} 
0, & \text{if } 0 < x < a, 0 < y < a, 0 < z < a \\
\infty, & \text{otherwise}
\end{cases} \]

(a) Find the eigenenergies and eigenstates. [10 points]
(b) Let us now introduce a perturbation

\[ V'(x, y, z) = \begin{cases} 
V_0, & \text{if } 0 < x < a/2, 0 < y < a/2, 0 < z < a \\
0, & \text{otherwise}
\end{cases} \]

Find the matrix form of \( V' \) between the first excited states. [20 points]
(c) Calculate then new eigenenergies and eigenvectors in terms of a and b. [10 points]
Problem 4. Consider a particle of mass \(m\) and charge \(q\) in a three-dimensional harmonic oscillator described by the Hamiltonian

\[
H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2
\]

with \(p = (p_x, p_y, p_z)\) and \(r = (x, y, z)\).

(a) Show that the eigenstates of \(H_0\) are eigenstates of angular momentum \(L_z\). [8 points]

Similarly one can show that the eigenstates of \(H_0\) may also be chosen as eigenstates of angular momentum \(L_x\) and \(L_y\), so that the eigenstates of \(H_0\) may be labeled \(|n, \ell, m_z\rangle\), where \(E_n = \hbar\omega(n + \frac{3}{2})\) with \(n = 0, 1, 2, \ldots\) is the eigenvalue of \(H_0\), \(\hbar^2\ell(\ell + 1)\) is the eigenvalue of \(L^2\) and \(m_z\hbar\) is the eigenvalue of \(L_z\). We assume that at time \(t = -\infty\) the oscillator is in its ground state, \(|0, 0, 0\rangle\). It is then acted upon by a spatially uniform but time dependent electric field

\[
\mathcal{E}(t) = \mathcal{E}_0 \exp(-t^2/\tau^2)\hat{z}
\]

(where \(\mathcal{E}_0\) and \(\tau\) are constant).

(b) Show that, to first order in the perturbation, the only possible excited state the oscillator could end up in is the \(|1, 1, 0\rangle\) state. [8 points]

(c) What is the probability for the oscillator to be found in this excited state at time \(t = \infty\)? [Note that \(\int_{-\infty}^{\infty} \exp[-(x - c)^2] dx = \sqrt{\pi}\) for any complex constant \(c\).] [8 points]

(d) The probability you obtain should vanish for both \(\tau \to 0\) and \(\tau \to \infty\). Explain briefly why this is the case. [8 points]

(e) If instead the oscillator was in the \(|1, 1, 0\rangle\) state at time \(t = -\infty\), show that the probability that it ends up in the ground state at time \(t = \infty\) is identical to what was found in part c). [8 points]
NIU Physics PhD Candidacy Exam – Fall 2012 – Quantum Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

**Problem 1.** A particle of mass $m$ moves in three dimensions in an attractive potential that is concentrated on a spherical shell:

$$V(r) = \begin{cases} -V_0a \delta(r - a) & (\text{for } r = a), \\ 0 & (\text{for } r \neq a), \end{cases}$$

where $V_0$ is a positive constant with units of energy, and $a$ is a fixed radius and $r$ is the radial spherical coordinate. Consider the lowest bound state of this system with wavefunction denoted $\psi(r) = R(r)/r$ and energy $E$. Write your answers below in terms of $\beta \equiv \sqrt{-2mE/\hbar^2}$ and $a, m, V_0$.

(a) Find the Schrödinger differential equation for $R(r)$. [6 points]

(b) By enforcing proper behavior of the wavefunction at $r = 0$ and $r = \infty$ and $r = a$, find the energy of the ground state in terms of the solution to a transcendental equation. [26 points]

(c) Find the smallest value of $V_0$ such that there is a bound state. [8 points]

**Problem 2.** A particle experiences a one-dimensional harmonic oscillator potential. The harmonic oscillator energy eigenstates are denoted by $|n\rangle$ with $E_n = (n + 1/2)\hbar\omega$. The state is given by $|\psi(t)\rangle$. At $t = 0$, the state describing the particle is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

(a) Calculate $\langle E(t) \rangle = \langle \psi(t)|H|\psi(t)\rangle$. [13 points]

(b) Calculate $\langle x(t) \rangle = \langle \psi(t)|x|\psi(t)\rangle$. [14 points]

(c) Calculate the root mean squared deviation of $x(t)$. [13 points]
Problem 3. Consider an atomic $p$ electron ($l = 1$) which is governed by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

where

$$\hat{H}_0 = \frac{a}{\hbar} \hat{L}_z^2 \quad \text{and} \quad \hat{H}_1 = \sqrt{2c} \frac{\hbar}{\hbar} \hat{L}_x,$$

with $a > 0$.

(a) Determine the Hamiltonian $\hat{H}$ in matrix form for a basis $|l,m\rangle$. Restrict yourself to $l = 1$. You can use the formula

$$\hat{L}_\pm |l,m\rangle = \hbar \sqrt{(l \mp m)(l \pm m + 1)} |l, m \mp 1\rangle \quad \text{where} \quad \hat{L}_\pm = \hat{L}_x \pm \hat{L}_y.$$

[14 points]

(b) We want to treat $\hat{H}_1$ as a perturbation of $\hat{H}_0$. What are the energy eigenvalues and eigenstates of the unperturbed problem? [6 points]

(c) We assume $a \gg |c|$. Calculate the eigenvalues and eigenstates of $\hat{H}$ in second and first order of the perturbation $\hat{H}_1$, respectively. [10 points]

(d) This problem can also be solved exactly. Give the exact eigenvalues and eigenstates and show that they agree with the results obtained in (c). [10 points]

Problem 4. Given the general form of the spin-orbit coupling on a particle of mass $m$ and spin $\hat{S}$ moving in a central force potential $V(r)$ is as follows ($\hat{H}_{SO}$)

$$\hat{H}_{SO} = \frac{1}{2m^2c^2} \hat{L} \cdot \hat{S} \frac{1}{r} \frac{dV(r)}{dr}.$$

Assume an electron ($s = 1/2$) in the central force potential $V(r)$ of a spherically symmetric 3D simple harmonic oscillator,

(a) Evaluate $\langle \hat{H}_{SO} \rangle$. [20 points]

(b) What is the energy shift for those states with $\ell = 0$? [10 points]

(c) What are the possible $j$ for those states with $\ell = 1$? [5 points]

(d) Evaluate the energy shift for those states with $\ell = 1$ [5 points]
Problem 1.

Assume that the lowest-energy eigenfunction of the simple harmonic oscillator is approximated by

\[ u_0(x) = N \exp(-ax^2), \tag{1} \]

where \( a \) is the constant we want to determine and \( N \) is a normalization constant.

(a) Determine the normalization constant \( N \). See also the integrals at the end of the question. [8 points]
(b) Calculate the energy in terms of \( a \) and the angular frequency \( \omega \). [12 points]
(c) Determine \( a \) by minimizing the energy with respect to \( a \). [10 points]
(b) Determine the energy associated with the lowest eigenfunction. [10 points]

\[ \int_0^\infty e^{-ax^2} \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \quad \text{and} \quad \int_0^\infty x^2 e^{-ax^2} \, dx = \frac{1}{4a} \sqrt{\frac{\pi}{a}} \tag{2} \]

Problem 2.

A particle of mass \( m \) is confined to an infinitely high one-dimensional potential box of width \( L \). At the bottom of the box, there is a bump in the potential of height \( V_0 \) and width \( L/2 \), as shown.

(a) Find the ground-state wavefunction to first order in perturbation theory in \( V_0 \). [18 points]
(b) Find the ground-state energy to second order in perturbation theory in \( V_0 \). [16 points]
(c) What condition must hold for the perturbation expansion to make sense? Give the answer in terms of \( V_0, m, L, \) and \( \hbar \). [6 points]
Problem 3.

The first excited state of a three-dimensional isotropic harmonic oscillator (of natural angular frequency $\omega_0$ and mass $m$) is three-fold degenerate.

(a) Calculate to first order the energy splittings of the three-fold degenerate state due to a small perturbation of the form $H' = bxy$ where $b$ is a constant. [25 points]

(b) Give the first-order wavefunction of those three levels in terms of the wavefunctions of the unperturbed three-dimensional harmonic oscillator. [15 points]

Hint: For a one-dimensional harmonic oscillator

$$\langle n|x|n+1 \rangle = \sqrt{\frac{\hbar(n+1)}{2m\omega_0}}. \quad (3)$$

Problem 4.

(a) Consider a spherical potential well with

$$U(r) = 0 \quad r < a; \quad U(r) = \infty \quad r > a \quad (4)$$

Give the radial part of the Schrödinger equation for $r < a$. [10 points]

(b) We are now only interested in the wavefunction for $l = 0$. Give the general solution of the differential equation by trying a solution for the radial part of the wavefunction $R(r) = P(r)/r$. (Normalization is not necessary). [10 points]

(c) Determine the behavior of the wavefunction for $r \to 0$. What does the condition that the wavefunction should not diverge at $r = 0$ imply? [10 points]

(d) Determine the energy eigenvalues for $l = 0$. [10 points]
Problem 1. A spinless, non-relativistic particle of mass \( m \) moves in a three-dimensional central potential \( V(r) \), which vanishes for \( r \to \infty \). The particle is in an exact energy eigenstate with wavefunction in spherical coordinates:

\[
\psi(\vec{r}) = C r^n e^{-\alpha r} \sin(\phi) \sin(2\theta),
\]

where \( C \) and \( n \) and \( \alpha \) are positive constants.

(a) What is the angular momentum of this state? Justify your answer. [10 points]

(b) What is the energy of the particle, and what is the potential \( V(r) \)? [30 points]

Problem 2. Apply the variational principle to the anharmonic oscillator having the Hamiltonian

\[
\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + Cx^4. \tag{1}
\]

(a) Use as your trial wavefunction a form that is similar to the wavefunction for a harmonic oscillator:

\[
\psi(x) = \left( \frac{\lambda^2}{\pi} \right)^{1/4} e^{-\frac{x^2}{2}} \tag{2}
\]

to determine the variational constant \( \lambda \) that minimizes the expectation value of \( \hat{H} \). Note: \( \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a} \). [25 points]

(b) The ground state energy of the anharmonic oscillator using numerical methods is

\[
E_0 = 1.060 \left( \frac{\hbar^2}{2m} \right)^{2/3} C^{1/3}. \tag{3}
\]

What is the approximate ground state energy using the variational method? How does it compare to the numerical results? [15 points]
Problem 3. The eigenstates of the hydrogen atom are perturbed by a constant uniform electric field \( E \) which points along the \( z \) direction. The perturbation is given by

\[
H' = -eEz = -eEr \cos \theta. \tag{4}
\]

The eigenstates \( \psi_{n\ell m}(r) \) are \( n^2 \)-fold degenerate for a particular \( n \). The electric field lifts this degeneracy. Let us consider \( n = 2 \).

(a) Write down the hydrogen wavefunctions for \( n = 2 \) when \( H' = 0 \). [5 points]

(b) Only two matrix elements of \( H' \) between the different states for \( n = 2 \) are nonzero. Explain why the other diagonal and off-diagonal matrix elements are zero. [5 points]

(c) Evaluate the matrix elements by using the expressions for the wavefunctions below and in the formula sheet. You do not have to evaluate the integral over the radial coordinate \( r \) (or \( \rho = r/a_0 \), where \( a_0 \) is the Bohr radius). [15 points]

(d) Find the eigenenergies and eigenfunctions for the \( n = 2 \) levels in the electric field. [15 points]

Additional information:

For \( Z = 1 \) and \( n = 2 \), we have

\[
R_{20} = \frac{1}{(2a_0)^{3/2}}(2 - \rho)e^{-\rho/2} \quad \text{and} \quad R_{21} = \frac{1}{\sqrt{3}(2a_0)^{3/2}}\rho e^{-\rho/2}, \tag{5}
\]

with \( \rho = r/a_0 \), where \( a_0 \) is the Bohr radius.

Problem 4. A free electron is at rest in a uniform magnetic field \( \vec{B} = \hat{x}B \). The interaction Hamiltonian is

\[
H = k \vec{S} \cdot \vec{B},
\]

where \( k \) is a constant. At time \( t = 0 \), the electron’s spin is measured to be pointing in the \( +\hat{z} \) direction.

(a) What is the probability that at time \( t = T \) the electron’s spin is measured to point in the \(-\hat{z}\) direction? [25 points]

(b) What is the probability that at time \( t = T \) the electron’s spin is measured to point in the \( +\hat{x} \) direction? [15 points]
Problem 1.

(a) In general, what is the first order correction to the energy of a quantum state for a one dimensional system with a time independent perturbation given by $H'$? [8 points]

(b) Suppose in an infinite square well between $x = 0$ and $x = a$ the perturbation is given by raising one half of the floor of the well by $V_0$. What is the change in energy to the even and to the odd states? [16 points]

(c) Now suppose the perturbation is given by $\alpha \delta(x - a/2)$ where $\alpha$ is constant. What is the first order correction to the allowed energies for the even and odd states? [16 points]

Problem 2.

We consider scattering off a spherical potential well given by

$$V(r) = \begin{cases} -V_0 & r \leq a \\ 0 & r > a \end{cases}, \quad a > 0$$

The particle’s mass is $m$. We restrict ourselves to low energies, where it is sufficient to consider $s$ wave scattering (angular momentum $l = 0$).

(a) Starting from the Schrödinger equation for this problem, derive the phase shift $\delta_0$. [14 points]

(b) Calculate the total scattering cross section $\sigma$ assuming a shallow potential well ($a \sqrt{2mV_0/\hbar^2} \ll 1$). [10 points]

(c) Show that the same total scattering cross section $\sigma$ as in b) is also obtained when using the Born approximation. Note: part c) is really independent of parts a) and b). [16 points]
Problem 3.

For a quantum harmonic oscillator, we have the position $\hat{x}$ and momentum $\hat{p}_x$ operators in terms of step operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \quad \text{and} \quad \hat{p}_x = i\sqrt{\frac{m\hbar}{2}}(a^\dagger - a) \quad (1)$$

giving a Hamiltonian $H = \hbar\omega(a^\dagger a + \frac{1}{2})$.

(a) The eigenstates with energy $(n + \frac{1}{2})\hbar\omega$ in bra-ket notation are $|n\rangle$. Express the eigenstates in terms of the step operators and the state $|0\rangle$ (no need to derive). [8 points]

(b) Show that the eigenstates $|1\rangle$ and $|2\rangle$ are normalized using the fact that $|0\rangle$ is normalized (or derive the normalization factor for those states, in case your result from (a) is not normalized). [8 points]

(c) Calculate the expectation values of $\langle \hat{x}^2 \rangle$ and $\langle \hat{p}_x^2 \rangle$ for the eigenstates $|n\rangle$. [8 points]

(d) Using the result from (c), show that the harmonic oscillator satisfies Heisenberg’s uncertainty principle (consider only eigenstates). [8 points]

(e) The term $H' = \gamma x^2$ is added to the Hamiltonian. Find the eigenenergies of $H + H'$. [8 points]

Problem 4.

(a) We want to study the spin-orbit coupling for an atomic level with $l = 2$. How will this level split under the interaction $\zeta \mathbf{L} \cdot \mathbf{S}$? Give also the degeneracies. [8 points]

(b) Show that for an arbitrary angular momentum operator (integer and half-integer), we can write

$$J_\pm |jm_j\rangle = \sqrt{(j \mp m_j)(j \pm m_j + 1)}|jm_j \pm 1\rangle \quad (2)$$

(take $\hbar = 1$) [Hint: Rewrite $J_\pm J_\mp$ in terms of $J^2$ and $J_z$.] [10 points]

(c) Since $m_j$ is a good quantum number for the spin-orbit coupling, we can consider the different $m_j$ values separately. Give the matrix for $\zeta \mathbf{L} \cdot \mathbf{S}$ in the $|lm, \frac{1}{2}\sigma\rangle$ basis with $\sigma = \pm \frac{1}{2}$ for $m_j = 3/2$. Find the eigenvalues and eigenstates of this matrix. [12 points]

(d) Write down the matrix for the spin-orbit coupling in the $|jm_j\rangle$ basis for $m_j = 3/2$. [10 points]
Problem 1.

We consider a spinless particle with mass $m$ and charge $q$ that is confined to move on a circle of radius $R$ centered around the origin in the $x$-$y$ plane.

(a) Write down the Schrödinger equation for this particle and solve it to find the eigenenergies and corresponding normalized eigenfunctions. Are there degeneracies? [10 points]

(b) This system is perturbed by an electric field $E$ pointing along the $x$ axis. To lowest nonvanishing order in perturbation theory, find the corrections to the eigenenergies of the system. [10 points]

(c) What are the corrections to the eigenfunctions due to the field $E$ in lowest nonvanishing order? [10 points]

(d) Next we consider instead of the electric field $E$ the effect of a magnetic field $B$ pointing along the $z$ axis. Evaluate to lowest nonvanishing order in perturbation theory the corrections to the eigenenergies of the system. [10 points]

Problem 2.

Let us consider two spins $S$ and $S'$ with $S = S' = \frac{1}{2}$. The $z$ components of the spin are $S_z = \pm \frac{1}{2}$ and $S'_z = \pm \frac{1}{2}$. We can define a basis set as $|SS_z, S'z_z\rangle$ (or simplified $|S_z, S'_z\rangle$). The spins interact with each other via the interaction

$$H = T S \cdot S', \quad (1)$$

where $T$ is a coupling constant. $S$ and $S'$ work on the spins $S$ and $S'$, respectively.

(a) Rewrite the interaction in terms of $S_z$, $S'_z$ and step up and down operators $S_{\pm}$ and $S'_{\pm}$. [10 points]

(b) Find the eigenvalues of $H$ when the spins are parallel. [10 points]

(c) Find the eigenvalues of $H$ for $S_z + S'_z = 0$. [13 points]

(d) Give a physical interpretation of the eigenenergies and eigenstates of $H$. [7 points]
**Problem 3.**
Given a one-dimensional harmonic oscillator with Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \]  

and a wavefunction which is a mixture of the \( n = 0 \) and \( n = 1 \) states

\[ \psi(x) = \frac{1}{\sqrt{5}}(u_0(x) - 2u_1(x)), \]

where \( u_0 \) and \( u_1 \) are the normalized eigenfunctions of the lowest two energy states. Note that \( a^\pm = (\pm ip_x + m\omega x)/\sqrt{2\hbar m\omega} \).

(a) draw \( \psi(x) \). [5 points]
(b) what is \( \langle E \rangle \) in terms of \( m \) and \( \omega \)? [8 points]
(c) what are \( \langle x \rangle, \langle x^2 \rangle \) and \( \Delta x \)? [17 points]
(d) what is \( \langle p \rangle \)? [10 points]

**Problem 4.**
The (unnormalized) eigenfunctions for the lowest energy eigenvalues of a one-dimensional simple harmonic oscillator (SHO) are

\[ \psi_0(x) = e^{-x^2/\alpha^2}, \quad \psi_1(x) = \frac{x}{\alpha} e^{-x^2/\alpha^2}, \quad \psi_2(x) = \left( 1 - \frac{4x^2}{\alpha^2} \right) e^{-x^2/\alpha^2}, \]

\[ \psi_3(x) = \left( \frac{3x}{\alpha} - \frac{4x^3}{\alpha^3} \right) e^{-x^2/\alpha^2}, \quad \psi_4(x) = \left( 3 - \frac{24x^2}{\alpha^2} + \frac{16x^4}{\alpha^4} \right) e^{-x^2/\alpha^2}. \]

Now consider an electron in “half” a one-dimensional SHO potential (as sketched below)

\[ V(x) = \begin{cases} Kx^2 & x > 0 \\ \infty & x \leq 0 \end{cases} \]  

(a) Sketch the ground and first excited state for this new potential. [6 points]

(b) Write the normalized wave function for the ground state in terms of the electron mass \( m \) and the oscillator frequency \( \omega \) (corresponding to the spring constant \( K \)). [6 points]
(c) What are the energy eigenvalues for the potential \( V(x) \)? [6 points]
(d) Now we add a constant electric field \( E \) in \( x \) direction. Use first-order perturbation theory to estimate the new ground state energy. [8 points]
(e) We go back to \( E = 0 \). Now we add a second electron. Ignoring the Coulomb interaction between the electrons, write the total energy and the new two-particle wave function, assuming that the electrons are in a singlet spin state with the lowest possible energy. (You can ignore wave function normalization now.) [7 points]
(f) Repeat part (e) assuming that the electrons are instead in a triplet spin state with the lowest possible energy. [7 points]
Problem 1. In many systems, the Hamiltonian is invariant under rotations. An example is the hydrogen atom where the potential $V(r)$ in the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(r),$$

depends only on the distance to the origin.

An infinitesimally small rotation along the $z$-axis of the wavefunction is given by

$$R_{z,d\varphi}\psi(x, y, z) = \psi(x - yd\varphi, y + xd\varphi, z),$$

(a) Show that this rotation can be expressed in terms of the angular momentum component $L_z$. [10 points]

(b) Starting from the expression of $L_z$ in Cartesian coordinates, show that $L_z$ can be related to the derivative with respect to $\varphi$ in spherical coordinates. Derive the $\varphi$-dependent part of the wavefunction corresponding to an eigenstate of $L_z$. [10 points]

(c) Show that if $R_{z,d\varphi}$ commutes with the Hamiltonian, then there exist eigenfunctions of $H$ that are also eigenfunctions of $R_{z,d\varphi}$. [10 points]

(d) Using the fact that $L_i$ with $i = x, y, z$ commute with the Hamiltonian, show that $L^2$ commutes with the Hamiltonian. [10 points]

Problem 2. To an harmonic oscillator Hamiltonian

$$H = \hbar \omega a^\dagger a,$$

we add a term

$$H' = \lambda (a^\dagger + a).$$

This problem is known as the displaced harmonic oscillator. It can be diagonalized exactly by adding a constant (let us call it $\Delta$) to the step operators.

(a) Express the constant $\Delta$ in terms of $\hbar \omega$ and $\lambda$. [8 points]

(b) The energies are shifted by a constant energy. Express that energy in terms of $\hbar \omega$ and $\lambda$. [8 points]

(c) Express the new eigenstates $|\tilde{n}\rangle$ in terms of the displaced oscillator operator $\tilde{a}^\dagger$. [8 points]

(d) Calculate the matrix elements $\langle \tilde{n} | 0 \rangle$. [8 points]

(e) An harmonic oscillator is in the ground state of $H$. At a certain time, the Hamiltonian suddenly changes to $H + H'$. Plot the probability and change in energy for the final states $|\tilde{n}\rangle$ for $\tilde{n} = 0, \cdots, 5$ for $\Delta = 2$. [8 points]
Problem 3. We consider scattering off a spherical potential well given by

\[ V(r) = \begin{cases} 
-V_0 & r \leq a \\
0 & r > a
\end{cases} \]

The particles’ mass is \( m \). We restrict ourselves to low energies, where it is sufficient to consider \( s \) wave scattering (angular momentum \( l = 0 \)).

(a) Starting from the Schrödinger equation for this problem, derive the phase shift \( \delta_0 \). [14 points]

(b) Calculate the total scattering cross section \( \sigma \) assuming a shallow potential well \((a\sqrt{2mV_0/\hbar^2} \ll 1)\). [10 points]

(c) Show that the same total scattering cross section \( \sigma \) as in b) is also obtained when using the Born approximation. Note: part c) is really independent of parts a) and b). [16 points]

Problem 4. The normalized wavefunctions for the \( 2s \) and \( 2p \) states of the hydrogen atom are:

\[ \psi_{2s} = \frac{1}{\sqrt{32\pi a^3}} (N - r/a) e^{-r/2a} \]
\[ \psi_{2p,0} = \frac{1}{\sqrt{32\pi a^3}} (r/a) e^{-r/2a} \cos \theta \]
\[ \psi_{2p,\pm 1} = \frac{1}{\sqrt{64\pi a^3}} (r/a) e^{-r/2a} \sin \theta e^{\pm i\phi} \]

where \( a \) is the Bohr radius and \( N \) is a certain rational number.

(a) Calculate \( N \). (Show your work; no credit for just writing down the answer.) [10 points]

(b) Find an expression for the probability of finding the electron at a distance greater than \( a \) from the nucleus, if the atom is in the \( 2p, +1 \) state. (You may leave this answer in the form of a single integral over one variable.) [10 points]

(c) Now suppose the atom is perturbed by a constant uniform electric field \( \vec{E} = E_0 \hat{z} \). Find the energies of the \( 2s \) and \( 2p \) states to first order in \( E_0 \). [20 points]
Problem 1.

Consider the effects of the hyperfine splitting of the ground state of the Hydrogen atom in the presence of an external magnetic field \( \vec{B} = B_0 \hat{z} \). Let the electron spin operator be \( \vec{S} \) and the proton spin operator be \( \vec{I} \), and call the total angular momentum operator \( \vec{J} = \vec{S} + \vec{I} \). Then the Hamiltonian for the system is:

\[
H = \frac{E_\gamma}{\hbar^2} \vec{S} \cdot \vec{I} + 2 \frac{\mu_B}{\hbar} \vec{B} \cdot \vec{S},
\]

where \( E_\gamma \) is the energy of the famous 21 cm line and \( \mu_B \) is the Bohr magneton. The states of the system may be written in terms of angular momentum eigenstates of \( S_z, I_z \) or \( J^2, J_z \), so clearly label which basis you are using in each of your answers.

(a) In the limit that \( B_0 \) is so large that \( E_\gamma \) can be neglected, find the energy eigenstates and eigenvalues. [12 points]

(b) In the limit that \( B_0 \) is so small that it can be neglected, find the energy eigenstates and eigenvalues. [15 points]

(c) Find the energy eigenvalues for general \( B_0 \), and show that the special limits obtained in parts (a) and (b) follow. [13 points]

Problem 2.

A quantum mechanical spinless particle of mass \( m \) is confined to move freely on the circumference of a circle of radius \( R \) in the \( x, y \) plane.

(a) Find the allowed energy levels of the particle, and the associated wavefunctions. [16 points]

(b) Now suppose the particle has a charge \( q \) and is placed in a constant electric field which is also in the \( x, y \) plane. Calculate the shifts in energy levels to second order in the electric field, treated as a perturbation. [16 points]

(c) Show that the degeneracies are not removed to any order in the electric field treated as a perturbation. [8 points]
Problem 3.

Given a 2D harmonic oscillator with Hamiltonian
\[ H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + kmxy \]  
(2)

(a) How does \( \langle x \rangle \) change with time, that is determine \[ d\langle x \rangle/dt \] ? [10 points]
(b) For \( k = 0 \), what are the energies of the ground state and first and second excited states? What are the degeneracies of each state? [15 points]
(c) For \( k > 0 \), using first order perturbation theory, what are the energy shifts of the ground state and the first excited states? [15 points]

Problem 4.

In many systems, the Hamiltonian is invariant under rotations. An example is the hydrogen atom where the potential \( V(r) \) in the Hamiltonian
\[ H = \frac{p^2}{2m} + V(r), \]  
(3)
depends only on the distance to the origin.

An infinitesimally small rotation along the z-axis of the wavefunction is given by
\[ R_{z,d\varphi}\psi(x, y, z) = \psi(x - yd\varphi, y + xd\varphi, z), \]  
(4)

(a) Show that this rotation can be expressed in terms of the angular momentum component \( L_z \). [10 points]
(b) Starting from the expression of \( L_z \) in Cartesian coordinates, show that \( L_z \) can be related to the derivative with respect to \( \varphi \) in spherical coordinates. Derive the \( \varphi \)-dependent part of the wavefunction corresponding to an eigenstate of \( L_z \). [10 points]
(c) Show that if \( R_{z,d\varphi} \) commutes with the Hamiltonian, then there exist eigenfunctions of \( H \) that are also eigenfunctions of \( R_{z,d\varphi} \). [10 points]
(d) Using the fact that \( L_i \) with \( i = x, y, z \) commute with the Hamiltonian, show that \( L^2 \) commutes with the Hamiltonian. [10 points]
Problem 1.

The diagram shows the six lowest energy levels and the associated angular momenta for a spinless particle moving in a certain three-dimensional central potential. There are no “accidental” degeneracies in this energy spectrum. Give the number of nodes (changes in sign) in the radial wave function associated with each level. Justify your answer.

Problem 2.

Assume that the mu-neutrino $\nu_\mu$ and the tau-neutrino $\nu_\tau$ are composed of a mixture of two mass eigenstates $\nu_1$ and $\nu_2$. The mixing ratio is given by

$$
\begin{pmatrix}
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix}
\quad (1)
$$

In free space, the states $\nu_1$ and $\nu_2$ evolve according to

$$
\begin{pmatrix}
|\nu_1(x,t)\rangle \\
|\nu_2(x,t)\rangle
\end{pmatrix} = e^{ipx/\hbar} \begin{pmatrix}
e^{-iE_1t/\hbar}|\nu_1(0)\rangle \\
e^{-iE_2t/\hbar}|\nu_2(0)\rangle
\end{pmatrix}
\quad (2)
$$

Show that the transition probability for a mu-neutrino into a tau-neutrino is given by

$$
P(\mu \rightarrow \tau) = \sin^2(2\theta) \sin^2 \left( \frac{E_2 - E_1}{2\hbar} t \right).
\quad (3)$$
Problem 3.
Let the potential $V = 0$ for $r < a_0$ (the Bohr radius) and $V = \infty$ for $r > a_0$. $V$ is a function of $r$ only.

a) What is the energy of an electron in the lowest energy state of this potential?
b) How does that compare to the energy of the $1s$ state of Hydrogen?
c) What is the approximate energy of the lowest energy state with angular momentum greater than 0 (you can leave this result in integral form)?

Problem 4.
The Hamiltonian for a two-dimensional harmonic oscillator is given by

$$H_0 = (a^\dagger a + b^\dagger b + 1)\hbar \omega, \quad (4)$$

with the coordinates given by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \quad \text{and} \quad y = \sqrt{\frac{\hbar}{2m\omega}}(b^\dagger + b). \quad (5)$$

a) Give the eigenvalues of this Hamiltonian.

The Hamiltonian is perturbed by

$$H' = \alpha xy, \quad (6)$$

where $\alpha$ is a small constant.

b) Express $H'$ in terms of the operators $a$ and $b$ and their conjugates.

c) Using degenerate perturbation theory, show how the eigenvalues are changed by $H'$ for the states with eigenenergy $2\hbar \omega$ for $H_0$. 
Problem 1

We consider a system made of the orthonormalized spin states $|+\rangle$ and $|\rangle$, where $S_z|\pm\rangle = \pm(h/2)|\pm\rangle$. Initially both of these states are energy eigenstates with the same energy $\epsilon$.

a) An interaction $V$ couples these spin states, giving rise to the matrix elements

$$\langle 1|V|1 \rangle = \langle 2|V|2 \rangle = 0 \quad \text{and} \quad \langle 1|V|2 \rangle = \Delta$$

Give the Hamiltonian $H$ of the interacting system.

b) Determine the energy eigenvalues of $H$.

c) Show that the states $|A\rangle$ and $|B\rangle$

$$|A\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle + \frac{\Delta^*}{|\Delta|} |\rangle \right) \quad \text{and} \quad |B\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle - \frac{\Delta^*}{|\Delta|} |\rangle \right)$$

are orthonormalized eigenstates of the interacting system.

d) Determine the time evolution of a state $|\psi(t)\rangle$ for which $|\psi(t = 0)\rangle = |+\rangle$.

e) For the state $|\psi(t)\rangle$, calculate the probability that a measurement of $S_z$ at time $t$ yields $\pm h/2$.

Problem 2

A particle of mass $m$ is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and normalized wave function.
Problem 3.

Given a two-dimensional oscillator with Hamiltonian
\[ H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2) + kmxy, \] (1)

a) What is the time dependence of \(d\langle x \rangle/dt\)?

b) What is the time dependence of \(d\langle p_x \rangle/dt\)?

c) For \(k = 0\), what are the energies of the ground state and the first and second excited states? What are the degeneracies of each state?

d) For \(k > 0\), using first-order perturbation theory, what are the energy shifts of the ground state and the first excited states?

Problem 4.

a) We want to study the spin-orbit coupling for a level with \(l = 3\). How do you expect that this level will split under the interaction \(\zeta \mathbf{L} \cdot \mathbf{S}\). Give also the degeneracies.

b) Show that for an arbitrary angular momentum operator (integer and half-integer), we can write
\[ J_\pm |jm_j\rangle = \sqrt{(j \mp m_j)(j \pm m_j + 1)}|jm_j\rangle \] (2)
(take \(\hbar = 1\)).

c) Since \(m_j\) is a good quantum number for the spin-orbit coupling, we can consider the different \(m_j\) values separately. Give the matrix for \(\zeta \mathbf{L} \cdot \mathbf{S}\) in the \(|lm, \pm \sigma\rangle\) basis with \(\sigma = \pm \frac{1}{2}\) for \(m_j = 5/2\). Find the eigenvalues and eigenstates of this matrix.

d) Write down the matrix for the spin-orbit coupling in the \(|jm_j\rangle\) basis for \(m_j = 5/2\).

e) Obtain the same eigenstates as in question c) by starting for the \(m_j = 7/2\) state using the step operators.
Problem 1.

Consider a particle with mass $m$ confined to a three-dimensional spherical potential well

$$V(r) = \begin{cases} 
0, & r \leq a \\
V_0, & r > a
\end{cases} \quad (1)$$

a) Give the Schrödinger equation for this problem.

b) Determine the explicit expressions for the ground state energy and the ground state wave function in the limit $V_0 \to \infty$.

c) For the more general case $0 < V_0 < \infty$, determine the transcendental equation from which we can obtain the eigenenergies of the particle for angular momentum $l = 0$.

d) Which condition must be fulfilled such that the transcendental equation derived in c) can be solved? (Hint: consider a graphical solution of the equation.) Compare this result with a particle in a one-dimensional rectangular well of depth $V_0$.

Problem 2.

The ground state energy and Bohr radius for the Hydrogen atom are

$$E_1 = -\frac{\hbar^2}{2ma_B^2}, \quad a_B = \frac{4\pi\varepsilon_0\hbar^2}{e^2m}. \quad (2)$$

a) Calculate the ground state energy (in eV) and Bohr radius (in nm) of positronium (a hydrogen-like system consisting of an electron and a positron).

b) What is the degeneracy of the positronium ground state due to the spin? Write down the possible eigenvalues of the total spin together with the corresponding wavefunctions.

c) The ground state of positronium can decay by annihilation into photons. Calculate the energy and angular momentum released in the process and prove there must be at least two photons in the final state.
Problem 3.

A particle of mass $m$ moves in one dimension inside a box of length $L$. Use first order perturbation theory to calculate the lowest order correction to the energy levels arising from the relativistic variation of the particle mass. You can assume that the effect of relativity is small. Note that the free particle relativistic Hamiltonian is $\hat{H}_{\text{rel}} = \sqrt{m^2c^4 + \hat{p}^2c^2} - mc^2$.

Problem 4.

a) Prove the variational theorem that states that for any arbitrary state

$$\langle \psi | \hat{H} | \psi \rangle \geq E_0. \quad (3)$$

b) Consider the Hamiltonian for a particle moving in one dimension

$$\hat{H} = \frac{\hat{p}^2}{2m} + V_0 \left( \frac{x}{a} \right)^6, \quad (4)$$

where $m$ is mass, $a$ a length scale, and $V_0$ an energy scale. Is the wavefunction

$$\psi(x) = C(x^2 - a^2)e^{-\left(x/d\right)^4}, \quad (5)$$

where $C$ is a normalization constant, $d$ an adjustable parameter with dimensions of length, a good choice for the variational approximation to the ground state? Why or why not?

c) Depending on the answer of the previous part: If it is a good choice, make a rough order-of-magnitude estimate of the optimal choice for $d$. If it is not a good choice, propose a better variational wavefunction, including an estimate of the length scale.
Problem 1.

The spin-orbit coupling of an electron of angular momentum \( l \) and spin \( s = \frac{1}{2} \) is described by the Hamiltonian

\[
H = \lambda \mathbf{l} \cdot \mathbf{s},
\]

where \( \lambda \) is the spin-orbit coupling parameter.

(a) Write down the matrix \( H \) and diagonalize it to show that the state is split into two states with total angular momentum \( j = l \pm \frac{1}{2} \). Find the eigenenergies.

(b) Show that the eigenenergies can also be determined using the relation \( j = l + s \).

[The raising and lowering operators for \( l \) are \( L_+|l,m_i\rangle = \sqrt{(l-m_i)(l+m_i+1)}|l,m_i+1\rangle \) and \( L_-|l,m_i\rangle = \sqrt{(l+m_i)(l-m_i+1)}|l,m_i-1\rangle \) and similarly for \( s \).]

Problem 2.

(a) Write down the nonrelativistic Hamiltonian for a Helium atom with two electrons.

(b) Write down the ground state wavefunction (include the spin part) and give the ground state energy \( E_0 \) (in eV) in the absence of electron-electron interactions.

(c) Write down the matrix element for the lowest-order correction \( E_1 \) to the ground-state energy due to the electron-electron interaction. The 1s orbital is given by

\[
\varphi_{100} = \frac{1}{\sqrt{8\pi}} \left( \frac{2Z}{a_0} \right)^{3/2} e^{-Zr/a_0}
\]

The matrix element can be evaluated giving \( \frac{5}{3} \frac{Ze^2}{4\pi\varepsilon_0a_0} \).

(d) Find the ratio of the correction \( E_1 \) to \( E_0 \).

The Rydberg constant is \( R = \frac{Z^2\hbar^2}{2ma_0^2} \) and the Bohr radius is \( a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2} \).
Problem 3.

Two identical spin 1/2 fermions described by position coordinates $\vec{r}_i$ ($i = 1, 2$) are bound in a three-dimensional isotropic harmonic oscillator potential

$$V(\vec{r}_i) = \frac{1}{2}m\omega^2 r_i^2.$$  \hfill (3)

(a) Write the wave functions of the system in terms of the single-particle spin eigenstates and the one-dimensional harmonic oscillator wave functions, for each of the energy eigenstates up to and including energy $4\hbar\omega$.

(b) Assume that in addition there is a weak spin-independent interaction $V$ between the particles:

$$V(\vec{r}_1 - \vec{r}_2) = -\lambda \delta^{(3)}(\vec{r}_1 - \vec{r}_2)$$ \hfill (4)

Find the energies of the system correct to first order in $\lambda$ for each of the unperturbed states found in part (a). You may leave your results in terms of definite integrals over known functions.

Problem 4.

Consider normal 1-dimensional particle in box potential ($V(x) = \infty$ for $|x| > L/2$ and $V(x) = 0$ inside box. Two identical particles are confined to the box (assume only orbital degrees of freedom, ignore spin).

(a) What is the normalized unperturbed ground state for

- two identical bosons of mass $m$ confined in the box
- two identical fermions of mass $m$ confined in the box
- And what are the unperturbed ground state energies of the two cases?

(b) Now a perturbation is applied. A small rectangular bump appears in the box between $-a/2$ and $+a/2$. This perturbation is $V_{\text{pert}} = +|V_0|$ for $|x| < a/2$ and is zero otherwise.

Use first-order perturbation theory to obtain the new ground state energies for the two cases.
Problem 1.

A particle moves in a 1 dimensional potential described by an attractive delta function at the origin. The potential is:

\[ V(x) = -W\delta(x) \]

(a) Discuss and determine the wavefunctions valid for bound state solutions of this system.

(b) Show that there is only one bound state and determine its energy.

Problem 2.

In this problem, \( |0\rangle \), \( |n\rangle \) are the shorthand for the eigenstates of the 1 dimensional simple harmonic oscillator (SHO) Hamiltonian, with \(|0\rangle\), denoting the ground state. The \( \hat{a}^\dagger \) and \( \hat{a} \) are the SHO raising and lowering operators. (sometimes termed creation and destruction (annihilation) operators).

(a) Prove that the following state vector \( |z\rangle \) is an eigenstate of the lowering operator \( \hat{a} \) and that its eigenvalue is \( z \).

\[ |z\rangle = e^{z\hat{a}^\dagger}|0\rangle \]

The \( z \) is an arbitrary complex number. (Note, knowledge of the expansion of \( e^z \) will be useful.)

(b) Evaluate \( \langle z_1|z_2\rangle \), where \( z_1 \) and \( z_2 \) are arbitrary complex numbers, and use this result to normalize state \( |z\rangle \).

Problem 3.

Let the potential \( V = 0 \) for \( r < a_0 \) (the Bohr radius) and \( V = \infty \) for \( r > a_0 \). \( V \) is a function of \( r \) only.

(a) What is the energy of an electron in the lowest energy state of this potential?

(b) How does this compare to the kinetic energy of the 1s state of Hydrogen?

(c) What is the approximate energy of the lowest energy state with angular momentum greater than 0 (you can leave this result in integral form)?

Problem 4.

In a magnetic resonance experiment a specimen containing nuclei of spin \( I = \frac{1}{2} \) and magnetic moment \( \mu = h\gamma I \) is placed in a static magnetic field \( B_0 \) directed along the \( z \)-axis and a field \( B_1 \) which rotates in the \( xy \)-plane with angular frequency \( \omega \).
(a) Write down the Hamiltonian for the system.

(b) If the wave function is written

$$\psi(t) = c_+(t)\chi_1 + c_-(t)\chi_{-\frac{1}{2}}$$

where $\chi_\frac{1}{2}$ and $\chi_{-\frac{1}{2}}$ are the spin eigenfunctions, show that

$$i\frac{dc_+}{dt} = \frac{1}{2}\omega_0 c_+ + \frac{1}{2}\omega_1 c_- e^{-i\omega t}$$

and

$$i\frac{dc_-}{dt} = -\frac{1}{2}\omega_0 c_- + \frac{1}{2}\omega_1 c_+ e^{i\omega t}$$

and where $\omega_0 = \gamma B_0$ and $\omega_1 = \gamma B_1$. Assuming that the system starts in the state $\chi_{-\frac{1}{2}}$, i.e. $c_+(0) = 0$ and $c_-(0) = 1$, solve these equations to show that subsequently the probability that the system is in the state $\chi_\frac{1}{2}$ is

$$|c_+|^2 = \frac{\omega_1^2 \sin^2 \frac{1}{2} \left[(\omega - \omega_0)^2 + \omega_1^2\right] \frac{1}{2} t}{(\omega - \omega_0)^2 + \omega_1^2}$$
PROBLEM 1

In the presence of a magnetic field $\mathbf{B} = (B_x, B_y, B_z)$, the dynamics of the spin $1/2$ of an electron is characterized by the Hamiltonian $H = -\mu_B \mathbf{σ} \cdot \mathbf{B}$ where $\mu_B$ is the Bohr magneton and $\mathbf{σ} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli spin matrices.

(a) Give an explicit matrix representation for $H$.

In the following, we investigate the time-dependent two-component wave function $\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ characterizing the dynamics of the electron spin. (The orbital part of the electron dynamics is completely ignored.)

(b) We assume that for $t < 0$ the magnetic field $\mathbf{B}$ is parallel to the $z$ axis, $\mathbf{B}(t < 0) = (0, 0, B_z)$ and constant in time. From the time-dependent Schrödinger equation, calculate $\psi(t)$ such that $\psi(t = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) At $t = 0$ an additional magnetic field in $x$ direction is switched on so that we have $\mathbf{B}(t \geq 0) = (B_x, 0, B_z)$. Solve the time-dependent Schrödinger equation for $t \geq 0$ using the ansatz $\psi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega t + a_2 \sin \omega t \\ b_1 \cos \omega t + b_2 \sin \omega t \end{pmatrix}$

Hint: The boundary condition $\psi(t = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ simplifies the calculation of the frequency $\omega$ and the coefficients $a_1, a_2, b_1,$ and $b_2$.

Note also that in order to get a solution $\psi(t)$ valid for all times $t \geq 0$, we may split the coupled equations into equations proportional to $\sin \omega t$ and $\cos \omega t$.

(d) Verify the normalization condition $|a(t)|^2 + |b(t)|^2 = 1$.

(e) Interpret your result for $|b(t)|^2$ by considering the limiting cases $B_x \ll B_z$ and $B_x \gg B_z$.

PROBLEM 2

A particle experiences a one-dimensional harmonic oscillator potential. The harmonic oscillator energy eigenstates are denoted by $|n\rangle$ with $E_n = (n + 1/2)\hbar\omega$. At $t = 0$, the state describing the particle is $|\psi, t= 0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

(a) Calculate $\langle E(t) \rangle = \langle \psi, t | H | \psi, t \rangle$
(b) Calculate \( \langle x(t) \rangle = \langle \psi, t | x | \psi, t \rangle \).

(c) Calculate the root mean squared deviation of \( x(t) \).

**PROBLEM 3.**

Consider a system of two *distinguishable* particles of spin \( \hbar/2 \). All degrees of freedom other than spin are ignored. Let \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) be the vector operators for spins of the particles. The Hamiltonian of this system is

\[
\hat{H} = A \mathbf{s}_1 \cdot \mathbf{s}_2
\]

with \( A \) a constant.

(a) Determine the energy eigenvalues and the accompanying eigenstates of this system.

(b) A system is prepared so that particle 1 is spin up \( (s_{1,z} = \hbar/2) \) and particle 2 is spin down, \( (s_{1,z} = -\hbar/2) \). Express this wavefunction in terms of the eigenstates of the Hamiltonian.

**PROBLEM 4.**

Let us consider two orbital angular momenta \( L_1 = L_2 = 1 \) that interact via \( H = \alpha \mathbf{L}_1 \cdot \mathbf{L}_2 \). The basis set is denoted by \( |L_1 M_1, L_2 M_2\rangle \), where \( M_i \) is the \( z \) component of \( L_i \) with \( i = 1, 2 \).

(a) Calculate the matrix element \( \langle 11, 11 | H | 11, 11 \rangle \). Is this an eigenenergy (explain)?

(b) Calculate the matrix elements \( \langle 11, 10 | H | 11, 10 \rangle \), \( \langle 10, 11 | H | 10, 11 \rangle \), and \( \langle 11, 10 | H | 10, 11 \rangle \). Use these matrix elements to derive the eigenenergies and eigenfunctions for \( M_1 + M_2 = 1 \).

(c) An alternative way to derive the eigenenergies is to express \( \mathbf{L}_1 \cdot \mathbf{L}_2 \) in \( \mathbf{L}_1^2, \mathbf{L}_2^2 \), and \( \mathbf{L}^2 \) where \( \mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \). Derive this expression and determine the eigenenergies for all possible values of \( \mathbf{L} \).

**Note:**

\[
L_\pm |LM\rangle = \sqrt{(L \pm M)(L \pm M + 1)} |L, M \pm 1\rangle
\]
Additional material

For spin 1/2 particles, spin operators are $s_i = \frac{\hbar}{2} \sigma_i$

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
Problem 1.

Consider an atomic $p$ electron ($l = 1$) which is governed by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$ where

$$\hat{H}_0 = \frac{b}{\hbar} \hat{l}_z + \frac{a}{\hbar^2} \hat{l}_z^2$$

and

$$\hat{V} = \sqrt{2} \frac{c}{\hbar} \hat{L}_x$$

(a) Show that within the basis of $l = 1$ states, $|1, m\rangle$, where $m$ denotes the $z$ component of $l$, the Hamiltonian $\hat{H}$ reads

$$\hat{H} = \begin{pmatrix} a + b & c & 0 \\ c & 0 & c \\ 0 & c & a - b \end{pmatrix}$$

You may want to use the formula

$$\hat{L}_\pm |l, m\rangle = \hbar \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle$$

where $\hat{L}_\pm = \hat{L}_x \pm \hat{L}_y$.

(b) We want to treat $\hat{V}$ as a perturbation of $\hat{H}_0$. What are the energy eigenvalues and eigenstates of the unperturbed problem?

(c) We assume $|a \pm b| \gg |c|$. Calculate the eigenvalues and eigenstates of $\hat{H}$ in second and first order of the perturbation $\hat{V}$, respectively.

(d) Next we consider $a = b$, $|a| \gg |c|$. Calculate the eigenvalues $\hat{H}$ in first order of the perturbation $\hat{V}$.

Problem 2.

Consider a harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2.$$ 

Do the following algebraically, that is, without using wave functions.

(a) Construct a linear combination $|\psi\rangle$ of the ground state $|0\rangle$ and the first excited state $|1\rangle$ such that the expectation value $\langle x \rangle = \langle \psi | x | \psi \rangle$ is as large as possible.

(b) Suppose at $t = 0$ the oscillator is in the state constructed in (a). What is the state vector for $t > 0$? Evaluate the expectation value $\langle x \rangle$ as a function of time for $t > 0$.

(c) Evaluate the expectation value $\Delta^2 x = \langle (x - \langle x \rangle)^2 \rangle$ as a function of time for the state constructed in (a). You may use

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \quad \text{and} \quad p = \sqrt{\frac{\hbar m \omega}{2}} (a - a^\dagger),$$

where $a$ and $a^\dagger$ are the annihilation and creation operators for the oscillator eigenstates.
Problem 3.

If a hydrogen atom is placed in a strong magnetic field, its orbital and spin magnetic dipole moments precess independently about the external field, and its energy depends on the quantum numbers \( m_\lambda \) and \( m_s \) which specify their components along the external field direction. The potential energy of the magnetic dipole moments is

\[
\Delta E = -(\mu_\lambda + \mu_s) \cdot B
\]

(a) For \( n = 2 \) and \( n = 1 \), enumerate all the possible quantum states \((n, \lambda, m_\lambda, m_s)\).

(b) Draw the energy level diagram for the atom in a strong magnetic field, and enumerate the quantum numbers and energy (the energy in terms of \( E_1 \), \( E_2 \), and \( \mu_B B_z \)) of each component of the pattern.

(c) Examine in your diagram the most widely separated energy levels for the \( n = 2 \) state. If this energy difference was equal to the difference in energy between the \( n = 1 \) and the \( n = 2 \) levels in the absence of a field, calculate what the strength of the external magnetic field would have to be. (Note: Bohr magneton is \( \mu_B = 9.27 \times 10^{-24} \) Joule/Tesla ). In the lab, the strongest field we can produce is on the order of 100 Tesla — how does your answer compare to this value? (Note: 1 eV=1.602\times10^{-19} \) J).

(d) Using the dipole selection rules, draw all the possible transitions among the \( n = 2 \) and \( n = 1 \) levels in the presence of a magnetic field.

Problem 4.

A particle of mass \( m \) is in an infinite potential well perturbed as shown in Figure 1.

a) Calculate the first order energy shift for the \( n \)th eigenvalue due to the perturbation.

b) Calculate the 2nd order energy shift for the ground state.

Some useful equations:

\[
\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4} \sin 2x
\]

\[
\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4} \sin 2x
\]

\[
2 \int \sin ax \sin bx dx = \frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}
\]

\[
2 \int \sin ax \cos bx dx = -\frac{\cos(a-b)x}{a-b} + \frac{\cos(a+b)x}{a+b}
\]

\[
2 \int \cos ax \cos bx dx = \frac{\sin(a-b)x}{a-b} + \frac{\sin(a+b)x}{a+b}
\]
2005 Ph.D. Qualifier, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.
A particle of mass \(m\) in an infinitely deep square well extending between \(x = 0\) and \(x = L\) has the wavefunction

\[
\Psi(x,t) = A \left[ \sin \left( \frac{\pi x}{L} \right) e^{-iE_1t/\hbar} - \frac{3}{4} \sin \left( \frac{3\pi x}{L} \right) e^{-iE_3t/\hbar} \right],
\]

where \(A\) is a normalization factor and \(E_n = n^2\hbar^2/(8mL^2)\).

(a) Calculate an expression for the probability density \(|\Psi(x,t)|^2\), within the well at \(t = 0\).
(b) Calculate the explicit time-dependent term in the probability density for \(t \neq 0\).
(c) In terms of \(m\), \(L\), and \(\hbar\), what is the repetition period \(T\) of the complete probability density?

Problem 2.
Let us consider the spherical harmonics with \(l = 1\).

(a) Determine the eigenvalues for \(aL_z\), where \(a\) is a constant.

(b) Determine the matrix for \(L_x\) for the basis set \(|lm\rangle\) with \(l = 1\) using the fact that

\[
L_\pm |lm\rangle = \hbar \sqrt{(l \mp m)(l \pm m + 1)} |l, m \pm 1\rangle.
\]

(c) Determine the eigenvalues of \(aL_x\) for the states with \(l = 1\).

(d) Determine the matrix for \(L^2\) from the matrices for \(L_+, L_-\), and \(L_z\).

Problem 3.
Consider a two-dimensional harmonic oscillator

\[
H_0 = \hbar \omega_x a_x^\dagger a_x + \hbar \omega_y a_y^\dagger a_y
\]

with \(\hbar \omega_x \ll \hbar \omega_y\). The number of excited states is given by \(N = n_x + n_y\), where \(a_x^\dagger|n_x\rangle = \sqrt{n_x + 1}|n_x + 1\rangle\) and \(a_y^\dagger|n_y\rangle = \sqrt{n_y + 1}|n_y + 1\rangle\)

(a) Express the normalized state with \(N = 2\) with the lowest energy in terms of the step operators and the vacuum state \(|0\rangle\), i.e. the state with no oscillators excited.

The system is now perturbed by

\[
H_1 = K(a_x^\dagger a_y + a_y^\dagger a_x).
\]

(b) Calculate for the state found in (a): the correction in energy up to first order.

(c) Express the correction in energy up to second order.

(d) Give the lowest-order correction to the wavefunction.

**NOTE:** The correction term to the wavefunction is given by

\[
|\psi_{1n}\rangle = \sum_{m \neq n} \frac{\langle \psi_0^m | H_1 | \psi_0^0 \rangle}{E_n^0 - E_m^0} |\psi_m^0\rangle
\]
Problem 4.
A system has unperturbed energy eigenstates \(|n\rangle\) with eigenvalues \(E_n\) (for \(n = 0, 1, 2, 3\ldots\)) of the unperturbed Hamiltonian. It is subject to a time-dependent perturbation

\[ H_I(t) = \frac{\hbar A}{\sqrt{\pi \tau}} e^{-t^2/\tau^2} \]  

where \(A\) is a time-independent operator.

(a) Suppose that at time \(t = -\infty\) the system is in its ground state \(|0\rangle\). Show that, to first order in the perturbation, the probability that the system will be in its \(m\)th excited state \(|m\rangle\) (with \(m > 0\)) at time \(t = +\infty\) is:

\[ P_m = a |\langle m | A | 0 \rangle|^2 e^{-br^2(E_0 - E_m)} \]  

(b) Next consider the limit of an impulsive perturbation, \(\tau \to 0\). Find the probability \(P_0\) that the system will remain in its ground state. Find a way of writing the result in terms of only the matrix elements \(|0\rangle|A^2|0\rangle\) and \(|0\rangle|A|0\rangle\).

**Hint:** the time evolution of states to first order in perturbation theory can be written as

\[ |\psi(t)\rangle = \left[ e^{-i(t-t_0)H_0/\hbar} - \frac{i}{\hbar} \int_{t_0}^{t} dt' e^{-i(t-t')H_0/\hbar} H_I(t') e^{-i(t' - t_0)H_0/\hbar} \right] |\psi(t_0)\rangle \]  

where \(H_0\) is the unperturbed time-independent Hamiltonian.
2005 Ph.D. Qualifier, Quantum Mechanics
DO ONLY 3 OUT OF 4 OF THE QUESTIONS

Problem 1.
A non-relativistic particle with energy $E$ and mass $m$ is scattered from a weak spherically-symmetric potential:

$$V(r) = \begin{cases} A(1 - r/a) & \text{for } r < a \\ 0 & \text{for } r \geq a, \end{cases}$$

where $a$ and $A$ are positive constants, and $r$ is the distance to the origin.

1(a) In the Born approximation, for which the scattering amplitude is given by

$$f(k, k') = -\frac{m^2}{2\pi\hbar^2} \int dr V(r) e^{i r (k - k')} ,$$

find the differential cross-section for elastic scattering at an angle $\theta$. (You may leave your result in terms of a well-defined real integral over a single real dimensionless variable.)

1(b) Show that in the low-energy limit the total scattering cross-section is proportional to $a^n$ where $n$ is an integer that you will find.

Problem 2.
Two distinguishable spin-$1/2$ fermions of the same mass $m$ are restricted to move in one dimension, with their coordinates given by $x_1$ and $x_2$. They have an interaction of the form

$$V(x_1 - x_2) = -g^2 \delta(x_1 - x_2) \frac{(S_1 \cdot S_2 + 1)}{2} ,$$

where $S_1$ and $S_2$ are the vector spin operators with eigenvalues of each component normalized to $\pm 1/2$.

Discuss the spectrum of eigenvalues, and find the bound state wavefunctions and energy eigenvalues. Also, discuss how these results change if the particles are indistinguishable.

Problem 3.
The eigenfunction for the first excited spherically symmetric state of the electron in a Hydrogen atom is given by

$$\psi(r) = A(1 - Br)e^{-Br}$$

3a. Show that this satisfies the Schrödinger equation and deduce the value of the constant $B$.

3b. Determine the energy for this state.

3c. Solve for the value of $A$ and thus obtain the expectation value of the distance $r$ from the origin.

Assume

$$\int_0^{\infty} r^n e^{-\alpha r} dr = \frac{n!}{\alpha^{n+1}}$$

Problem 4.
4(a). Let us consider an atom that can couple to Einstein oscillators of energy $\hbar \omega$. We can assume that the energy of the atom is zero. The Hamiltonian for the oscillators is given by

$$H = \hbar \omega (a^\dagger a + \frac{1}{2}) ,$$

where $a^\dagger$ and $a$ are the step up and step down operators. For $t < 0$, there is no coupling between the atom and the oscillators. Since no oscillators are excited, the system is in the ground state $|0\rangle$. At $t = 0$, a perturbation is created (for example, the atom is ionized) giving a coupling between the atom and the oscillators

$$H' = C(a + a^\dagger)$$

Show that the total Hamiltonian for $t > 0$, $H + H'$, can be diagonalized by adding a constant shift to the step operators and determine the shift.
4(b). Express the energy eigenstates $|n\rangle$ of the full Hamiltonian $H + H'$ in $a^\dagger$ and $|0\rangle$.
4(c). What are the matrix elements $\langle n'|0\rangle$, where $|0\rangle$ is the lowest eigenstate of $H$.
4(d). Assume the spectrum resulting from the sudden switching on of the perturbation is given by

$$ I(E) = \sum_{n'} |\langle n'|0\rangle|^2 \delta(E - E_{n'}). \quad (9) $$

Discuss the spectrum and how the spectral line shape changes as a function of $\Delta E/\hbar\omega$. 

$$ I(E) = \sum_{n'} |\langle n'|0\rangle|^2 \delta(E - E_{n'}). \quad (9) $$
**Answers**

3(a)

\[
A \frac{1}{r^2} \frac{d}{dr} (r^2(-B)e^{-Br}) = A \left(-B^2 + B^2\right)e^{-Br} 
\]

\[ (10) \]

\[-A \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \left[Bre^{-Br}\right]\right) = -A \frac{1}{r^2} \frac{d}{dr} \left(r^2 Be^{-Br} - B^2 r^3 e^{-Br}\right) \]

\[ = -A \left(\frac{2B}{r} - B^2 - 3B^2 + B^3 r\right) e^{-Br} \]

\[ (11) \]

giving

\[
A \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr}\right) = A \left(B^2 - \frac{4B}{r}\right)(1 - Br)e^{-Br} = \left(B^2 - \frac{4B}{r}\right)\psi 
\]

\[ (13) \]

For the Schrödinger Equation

\[ -\frac{\hbar^2}{2m} \left(B^2 - \frac{4B}{r}\right)\psi - \frac{e^2}{r}\psi = E\psi. \]

\[ (14) \]

Therefore

\[ \frac{\hbar^2 B}{2mr} = \frac{e^2}{r} \quad \Rightarrow \quad B = \frac{mc^2}{2\hbar^2} = \frac{1}{2a_0} \]

\[ (15) \]

where \(a_0 = \hbar v^2/mc^2\) is the Bohr radius. 3(b)

\[
E = -\frac{\hbar^2}{2m} B^2 = -\frac{\hbar^2}{4 \times 2ma_0^2} = -\frac{13.6}{4} \text{ eV} 
\]

\[ (16) \]
Problem 1. (a) Planck’s radiation law is given by

\[ u_\omega = \frac{\omega^2}{e^{\frac{\hbar \omega}{k_B T}} - 1}. \]  

(1)

Show that the energy density \( u_\omega \) in terms of wavelength becomes

\[ u_\lambda = \frac{8\pi \hbar c}{\lambda^5} \frac{1}{\exp\left(\frac{\hbar c}{\lambda k_B T}\right) - 1}. \]  

(2)

(b) Find the wavelength for which the energy distribution is maximum (assume that \( \hbar c/\lambda k_B T \) is large enough, so that \( e^{-\hbar c/\lambda k_B T} \to 0 \)). The relation \( T\lambda = \text{constant} \) is known as Wien’s Law.

(c) Derive the Stefan-Boltzmann law from Planck’s law, using

\[ \int_0^\infty \frac{x^3}{e^x - 1} \, dx = \frac{\pi^4}{15} = 6.4938. \]  

(3)

Calculate the value of the Stefan-Boltzmann constant.

Problem 2. (a) Let us consider the harmonic oscillator whose Hamiltonian is given by

\[ H = (a^\dagger a + \frac{1}{2})\hbar \omega. \]  

(4)

By using

\[ a^\dagger a |n\rangle = n |n\rangle \]  

(5)

and the commutation relations of the operators

\[ [a, a^\dagger] = 1 \]  

(6)

show that the wavefunction \( a^\dagger |n\rangle \) is proportional to the wavefunction \( |n + 1\rangle \).

(b) The wavefunctions for the harmonic oscillator are given by

\[ |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \]  

(7)

Determine the constant for which \( a^\dagger |n\rangle = \text{constant} |n + 1\rangle \).

(c) We can also write \( ˆx \) and \( ˆp_x \) in terms of the creation and annihilation operators

\[ ˆx = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \quad \text{and} \quad ˆp_x = i \sqrt{\frac{m\hbar \omega}{2}} (a - a^\dagger). \]  

(8)

By determining the values of \( \langle ˆx^2 \rangle \) and \( \langle ˆp_x^2 \rangle \), show that Heisenberg’s uncertainty principle is satisfied.
Problem 3. (a) An electron is harmonically bound at a site. It oscillates in the $x$ direction. The solutions of this harmonic oscillator are

$$H = \hbar \omega (a^\dagger a + \frac{1}{2}).$$

We introduce a perturbation by an electric field created by a positive point charge at a distance $R \hat{x}$. Show that the disturbing potential can be written as (you can leave out the constant energy shift from the electric field)

$$H' = P (a + a^\dagger) \quad \text{with} \quad P = \frac{e^2}{4\pi \epsilon_0 R^2} \sqrt{\frac{\hbar}{2m\omega}}$$

when $R$ is much larger than the amplitude of the oscillation.

(b) Note that the Hamiltonian $H + H'$ is not diagonal, since different values of $n$ are coupled with each other. Show that the total Hamiltonian can be diagonalized by adding a constant shift to the step operators.

(c) What is the shift in energy as a result of the perturbation $H'$?

Problem 4. Two similar metals are separated by a very thin insulating layer along the plane $x = 0$. The potential energy is constant inside each metal; however, a battery can be used to establish a potential difference $V_1$ between the two. Assume that the electrons have a strong attraction to the material of the insulating layer which can be modeled as an attractive delta function at $x = 0$ for all values of $y$ and $z$. A sketch of the potential energy along the $x$ direction is shown in Figure 1. Here $S$ and $V_1$ are positive.

(a) Assume that the metals extend to infinity in the $y$ and $z$ directions. Write down the correct three-dimensional functional form for an energy eigenfunction of a state bound in the $x$ direction. Sketch its $x$ dependence.

(b) Find the maximum value of $V_1$ for which a bound state can exist. Express your answer in terms of $\hbar$, $m$, and $S$.

(c) Find the energy of the bound state in terms of $\hbar$, $m$, $S$, and $V_1$. Show that your answer is consistent with 4(b).
Fig 1:

\[ V(x) = V_1 \]

\[ V = 0 \]

\[ V = -S\delta(x) \]
Ph. D. Qualifying Exam

Quantum Mechanics

Do 3 out of 4 problems

Problem 1:

The eigenfunction for the lowest spherically symmetric state of the electron in a hydrogen atom is given by

$$\psi (r) = Ae^{-br}$$

(a) Sketch the radial probability distribution for this state.

(b) Find the value of $r$ for which the radial probability is a maximum. This gives the Bohr radius $a_0$.

(c) Show that $\psi (r)$ satisfies the Schrödinger equation, and deduce the value of the Bohr radius in terms of $\hbar$, $m$, and $e$. What is the ground state energy in terms of the Bohr radius?

(d) Determine the normalization constant $A$ in terms of the Bohr radius.

(e) Find the value of the expectation value of $r$.

(f) Find the value of the expectation value of the potential energy.

(g) Find the value of the expectation value of the kinetic energy.
Problem 2:

A quantum mechanical particle of mass \( m \) is constrained to move in a cubic box of volume \( a^3 \). The particle moves freely within the box.

(a) Calculate the pressure the particle exerts on the walls of the box when the particle is in the ground state.

(b) Suppose the volume of the box is doubled suddenly by moving one wall of the box outward. What is the probability distribution of the energy of the particle after the expansion has taken place?

(c) What is the expectation value of the energy after the expansion?
Problem 3:

The spin-orbit coupling in hydrogen gives rise to a term in the Hamiltonian of the form $A \mathbf{L} \cdot \mathbf{S} / \hbar^2$ where $A$ is a positive constant with the units of energy.

(a) Find the zero-field splitting (separation) of the $n = 3, \ell = 2$ energy level of hydrogen due to this effect.

(b) Assume that the hydrogen atom is in the lowest of the energy levels found in Part (a) and that it has the maximum possible value of $m_j$ consistent with that energy. Find the probability density associated with a measurement of the $z$-component of the electron spin, $m_s$. 
Problem 4:

The Hamiltonian for the harmonic oscillator in one dimension is

\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \]

(a) Add a perturbation of the form \( \gamma \hat{x}^2 \) to the Hamiltonian. The solution to this perturbed harmonic oscillator can still be solved exactly. Calculate the new exact eigenenergies (you can assume the solution of the harmonic oscillator to be known—there is no need to derive it again).

(b) Instead of the perturbation \( \gamma \hat{x}^2 \), add the perturbation \( \gamma \hat{\dot{x}} \) to the Hamiltonian. Find the exact eigenenergies of this Hamiltonian.

(c) Using time-independent perturbation theory, show that the first order corrections to the energy vanishes for the Hamiltonian in Part (b).

(d) Calculate the second order corrections, and show that they agree with your answer to Part (b) showing that the second order corrections gives the complete solution for this problem.
Information which may be useful:

\[
[\hat{x}, \hat{p}] = i\hbar
\]

\[
\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i \sqrt{\frac{1}{2\hbar m\omega}} \hat{p}
\]

\[
\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i \sqrt{\frac{1}{2\hbar m\omega}} \hat{p}
\]

\[
\hat{x} = \frac{\hbar}{2m\omega} (\hat{a}^\dagger + \hat{a})
\]

\[
\hat{p} = i \sqrt{\frac{\hbar m\omega}{2}} (\hat{a}^\dagger - \hat{a})
\]

\[
[\hat{a}, \hat{a}^\dagger] = 1
\]

\[
\hat{a} |n\rangle = \sqrt{n} |n-1\rangle
\]

\[
\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle
\]

\[
\hat{H} = \left( \hat{a} \hat{a}^\dagger + \frac{1}{2} \right) \hbar \omega
\]

\[
\hat{J}_x |j, m\rangle = (\hat{J}_x \pm i \hat{J}_y) |j, m\rangle = \hbar \sqrt{j(j+1)-m(m+1)} |j, m\pm1\rangle
\]

The Laplacian in Spherical Coordinates is:

\[
\nabla^2 F = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \phi^2}
\]

Useful Integrals:

\[
\int_0^n r^n e^{-\alpha r} dr = \frac{n!}{\alpha^{n+1}}
\]

\[
\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}
\]
Problem 1:

(a) Find $\sigma_x^2 \sigma_p^2$ for an eigenstate, $|n\rangle$, of a harmonic oscillator with natural frequency $\omega$.

An exact expression, not a lower bound, is desired. $\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle$ is the variance associated with a measurement of the position, and $\sigma_p^2 = \langle (p_x - \langle p_x \rangle)^2 \rangle$ is the variance associated with a measurement of the momentum.

(b) Compare your answer to that which would be found for a classical harmonic oscillator of the same energy but undetermined phase where

$$x(t) = x_0 \sin(\omega t + \phi)$$
$$p(t) = p_0 \cos(\omega t + \phi).$$
Problem 2:

Consider the “hydrogen atom problem” in two dimensions. The electron is constrained to move in a plane and feels a potential $V(r) = -Z\varepsilon^2/r$ due to a charge $Ze$ at the origin. (This mathematical model has a physically realizable analog in the physics of semiconductors.)

(a) Find the eigenfunctions and eigenvalues for the $z$-component of angular momentum

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar \frac{\partial}{\partial \phi}$$

(b) The time independent Schrödinger equation for this problem is

$$\left(-\frac{\hbar^2}{2\mu} \nabla^2 + V(r)\right)\psi(r,\phi) = E\psi(r,\phi)$$

where $\mu$ is the reduced mass. Show that it is satisfied by a $\psi$ which is a product of radial and angular functions: $\psi(r,\phi) = R(r) \Phi(\phi)$. Find $\Phi(\phi)$ and write down the equation determining $R(r)$.

(c) What condition must be satisfied in order for $R(r) = \alpha e^{-r_0} r^n$ to be a solution of the radial equation? When this condition is satisfied find $r_0$ and the associated energy eigenvalue $E$ in terms of $\hbar$, $\mu$, $Z$, and $e$. ($\alpha$ is a normalization constant which you need not find.)

(d) Let $R(r) = r^{-1/2} u(r)$. Find the equation which determines $u(r)$. Comment on the form of this equation.

(e) A complete solution of the problem would show that the total degeneracy of the $n$th bound state energy eigenvalue is $2n - 1$. Draw an energy level diagram in which the levels are separated into different angular momentum “ladders”. Indicate the degeneracy and number of radial nodes associated with each of these “sub-levels” for the lowest 4 values of $E$. 
Problem 3:

A quantum mechanical particle of mass \( m \) moves in one dimension in a potential consisting of two negative delta-function spikes, located at \( x = \pm a \):

\[
V(x) = -\lambda [\delta(x - a) + \delta(x + a)],
\]

where \( \lambda \) is a positive constant.

(a) Prove that the basis of bound state wave functions can be chosen so that they are each either even or odd under reflection \( x \to -x \).

(b) Derive a (transcendental) equation for the binding energy of an even bound state. By sketching the functions involved, show that there is one and only one even bound state for each value of \( \lambda \).

(c) Derive the transcendental equation for an odd bound state. Show that there is a minimum value of \( \lambda \) for there to be an odd bound state, and determine that value.
Problem 4:

An atom that is otherwise spherically symmetric has an electron with orbital angular momentum $\ell = 2$ and spin $s = 1/2$.

(a) Using the raising and lowering operator formalism for general angular momenta, e.g.

$$J_-|j, m\rangle = (J_x - i J_y)|j, m\rangle = \sqrt{(j + m)(j - m + 1)} \hbar |j, m - 1\rangle,$$

construct the properly normalized linear combinations of $|\ell, m_\ell, s, m_s\rangle$ eigenstates that have total angular momentum eigenvalues:

(i) $j = 5/2, \quad m_j = 5/2$
(ii) $j = 5/2, \quad m_j = 3/2$
(iii) $j = 3/2, \quad m_j = 3/2$.

(b) In an external magnetic field in the $z$ direction of magnitude $B$, the magnetic interaction Hamiltonian is:

$$H_{\text{mag}} = \frac{eB}{2mc} (L_z + 2S_z).$$

What energy corrections are induced for the states of part (a), for a weak field $B$?
PhD Qualifying Exam January 2003 – Quantum Mechanics
Choose any 3 of 4.

Problem 1. A system with three unperturbed states can be represented by the perturbed Hamiltonian matrix:

\[ H = \begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}, \]

where \( E_2 > E_1 \). The quantities \( a \) and \( b \) are to be regarded as perturbations that are of the same order, and are small compared with \( E_1, E_2, \) and \( E_2 - E_1 \).

(a) Find the exact energy eigenvalues of the system.
(b) Use the second-order non-degenerate perturbation theory to calculate the perturbed energy eigenvalues (assuming the two degenerate energies are very slightly different). Is this procedure correct?
(c) Use the second-order degenerate perturbation theory to find the energy eigenvalues. Compare the three results obtained.

Problem 2. Consider a particle of mass \( m \) moving in the potential well:

\[ V = \begin{cases} 0 & \text{if } 0 < x < a \\ \infty & \text{elsewhere} \end{cases} \]

(a) Solve the Schrödinger equation to find the energy eigenvalues \( E_n \) and the normalized wavefunctions.
(b) For the \( n \)th energy eigenstate with energy \( E_n \), compute the expectation values of \( x \) and \( (x - \langle x \rangle)^2 \), and show that the results can be written:

\[ \langle x \rangle = \frac{a}{2}, \]
\[ \langle (x - \langle x \rangle)^2 \rangle = c_1 a^2 - \frac{c_2}{m E_n}, \]

where \( c_1 \) and \( c_2 \) are constant numbers that your will find. Show that in the limit of large \( E_n \), the results agree with the corresponding classical results.
Problem 3. Consider a particle of mass \( m \) moving in the three-dimensional harmonic oscillator potential \( V = \frac{k}{2}(x^2 + y^2 + z^2) \).
(a) Write down the Schrödinger equation for the wavefunction in rectangular coordinates. Find the allowed energy eigenvalues.
(b) What is the degeneracy of the 3rd excited energy level?
(c) Suppose the wavefunctions are found in spherical coordinates in a basis consisting of eigenstates of energy, total angular momentum, and the \( \hat{z} \) component of angular momentum, of the form:
\[
\psi(r, \theta, \phi) = R(r)Y_{\ell m}(\theta, \phi),
\]
where \( Y_{\ell m}(\theta, \phi) \) are the spherical harmonics. Find the differential equation satisfied by \( R(r) \), for a given energy eigenvalue \( E \) and a given \( \ell \) and \( m \).

Problem 4. A particle of spin \( \frac{1}{2} \) is subject to the Hamiltonian:
\[
H = \frac{a}{\hbar} S_z + \frac{b}{\hbar^2} S^2_z + \frac{c}{\hbar} S_x
\]
where \( a, b, \) and \( c \) are constants.
(a) What are the energy levels of the system?
(b) In each of the energy eigenstates, what is the probability of finding \( S_z = +\hbar/2 \)?
(c) Now suppose that \( a = 0 \), and that at time \( t = 0 \), the spin is in an energy eigenstate with \( S_z = +\hbar/2 \). What is the probability of finding \( S_z = -\hbar/2 \) at any later time \( t \)?
(d) Under the same assumptions as part (c), what is the expectation value of \( S_y \) as a function of \( t \)?

\* \* \* 

Possibly useful information:

The Pauli matrices are
\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The Laplacian in spherical coordinates is
\[
\nabla^2 F = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial F}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial F}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F}{\partial \phi^2}.
\]