Modern and Statistical Physics. September 2009
Do 3 of 5 problems

1. The face-centered cubic (fcc) is the most dense and the simple cubic (sc) is the least dense of the three cubic Bravais lattices. Suppose identical solid sphere are distributed through space in such a way that their centers lie on the points of each of these three structures, and spheres on neighboring points just touch, without overlapping. (Such an arrangement of sphere is called a close-packing arrangement)
   a). Assuming that the spheres have unit density, show that the density of a set of close-packed spheres on each of the three structures (the “packing fraction”) is $\sqrt{2}\pi/6 \approx 0.74$ for fcc, $\sqrt{3}\pi/8 \approx 0.68$ for body-centered cubic (bcc), and $\pi/6 \approx 0.52$ for sc.
   b). Show that for a fcc Bravais lattice the free electron Fermi sphere for valence 3 extends beyond point W of the first Brillouin zone (see figure), so that the first Brillouin zone is completely filled [Hint: prove $k_F/\Gamma W = (1296/125\pi^2)^{1/6} = 1.008$]

2. Consider the reaction $\gamma + e \rightarrow \pi^+ + \pi^- + e$ with the electron initially free and at rest (with this frame designated as the lab frame).
   a) What is the minimum photon energy for this reaction to proceed?
   b) Assuming that the photon energy is the minimum determined in a), what is the velocity of either of the pions after the reaction?
   c) Assume the pion then decays to a muon and neutrino $\pi \rightarrow \mu + \nu$. In the lab frame, what is the maximum energy of the muon produced in the pion decay assuming the photon energy determined in a).
   Give your answers in terms of the pion, muon, and electron masses while setting the photon and neutrino masses to 0.

3. A 55 year old man can focus objects clearly from 100 cm to 300 cm. Representing the eye as a simple lens 2 cm from the retina,
   a) what is the focal length of the lens at the far point (focused at 300 cm)?
   b) what is the focal length of the lens at the near point (focused at 100 cm)?
   c) what strength lens (focal length) must he wear in the lower part of his bifocal eyeglasses to focus at 25 cm?
4. A smooth vertical tube having two different sections is open from both ends and equipped with two pistons of different areas. Each piston slides within its respective tube section. One mole of ideal gas is enclosed between the pistons. The pistons are connected by a non-stretchable rod. The outside air pressure is 1 atm. The total mass of the pistons is M. The cross sectional area of the larger upper piston \( A_1 \), and the lower piston \( A_2 \) are related by \( A_1 = A_2 + \Delta A \). Find the rise in the temperature of the gas between the pistons required to lift the piston assembly by a distance \( L \).

![Diagram of the setup with pistons and ideal gas]

5. Assume that the neutron density in a neutron star is 0.1/fm\(^3\) (that is 0.1 neutron per cubic Fermi). Assuming \( T=0 \), ignoring any gravitational forces, and using a Fermi gas model with uniform density determine
   a) the average energy of the neutrons
   b) the average energy of electrons if the electron density is 1% of the neutron density
   c) show two reactions, one which can convert a neutron to a proton and one which can convert a proton to a neutron
   d) determine the neutron to proton ratio. Hint, consider the Fermi energies of the neutrons, protons and electrons at equilibrium

Give answers in a) and b) in MeV using \( hc = 1240 \text{ MeVfm} \), the mass of the neutron = 1000 MeV/c\(^2\), and the mass of the electron = 0.5 MeV/c\(^2\). You will need to decide if the particles are relativistic or non-relativistic. The answer for d) can be given in terms of the particle masses.