1) The potential energy of gas molecules in a certain central field depends on the distance \( r \) from the field’s center as \( U(r) = \alpha r^2 \) where \( \alpha \) is a positive constant. The gas temperature is \( T \), the concentration of molecules at the center of the field is \( n_0 \).

a) Find the number of molecules \( dN \), located at the distances between \( r \) and \( r+dr \) from the center of the field.

b) Find the most probable distance separating the molecules from the center of the field.

c) How many times will the concentration of molecules in the center of the field will change if the temperature decreases by \( \eta \) (\( T_{\text{new}} = \eta \times T \)). (Hint, how does \( dN/N \), that is, the fraction of molecules located in spherical layer between \( r \) and \( r+dr \), behave at the center)

d) Find the number of molecules \( dN \), whose potential energy lies within the interval from \( U \) to \( U+dU \).

FORMULA SHEET for PROBLEM 1

\[
\int_0^\infty x^n e^{-x^2} \, dx = \begin{cases} 
\frac{\sqrt{\pi}}{2}, & n = 0 \\
\frac{1}{2}, & n = 1 \\
\frac{\sqrt{\pi}}{4}, & n = 2 \\
\frac{1}{2}, & n = 3 
\end{cases}
\]

\[
\int_0^\infty x^n e^{-x} \, dx = \begin{cases} 
1, & n = 0 \\
\frac{\sqrt{\pi}}{2}, & n = 1 \\
\frac{\sqrt{\pi}}{4}, & n = 2 \\
\frac{1}{2}, & n = 1 \\
2, & n = 2 
\end{cases}
\]
2) Consider a particle of mass $m$ undergoing Brownian motion in one dimension. The particle is under the influence of a viscous friction force $-m\beta v$, an oscillatory driving force $-ma\sin(\omega t)$ and a random force $mA(t)$. Its equations of motion are
\[
\dot{x} = v, \quad \dot{v} + \beta v = -a \sin(\omega t) + A(t).
\]
a) Suppose $A(t) = 0$ and $\langle A(t)A(\tau) \rangle = \alpha \delta(t - \tau)$, where $\alpha$ is a constant and $\delta(t)$ is the Dirac delta function. Suppose the initial conditions are $q(0) = q_0$, $v(0) = \alpha/\omega$. Find the mean speed $\langle v(t) \rangle$. Note:
\[
\int_0^t d\tau e^{\beta \tau} \sin(\omega \tau) = \frac{e^{\beta t}(\beta \sin(\omega t) - \omega \cos(\omega t)) + \omega}{\beta^2 + \omega^2}
\]
b) Evaluate (by whatever means you chose) $\langle v(t) \rangle$ in the limit $\beta \rightarrow 0$, and provide a physical interpretation of your result.

3) The density of states for a free particle of momentum $p$ confined within a box of volume $\Omega$ is given by
\[
\frac{dn}{dp} = \frac{\Omega p^2}{2\pi \hbar^3}
\]
a) Calculate the density of states per unit energy for a non-relativistic electron and for a massless neutrino. Assume each is confined within a box of volume $\Omega$ but otherwise free of interactions.
b) Assume that the transition probability for beta decay is dominated by the density of states term. Take the electron to be non-relativistic and the neutrino massless. In terms of the total decay energy, $E_{\text{tot}}$, calculate the most likely energy for the emitted beta particle?

4) An energetic proton strikes a proton at rest. A K$^+$ is produced. Write down a reaction for producing a K$^+$ showing all the final state particles. What is the minimum kinetic energy for the incoming proton to produce this final state (express in terms of the masses of the particles)?

5) An interstellar proton interacts with the 3 degree cosmic microwave background (CMB) and produces an electron-positron pair via $p + \text{photon} \rightarrow p + \text{electron} + \text{positron}$. Assume the CMB is monoenergetic with $E = kT = .002 \text{ eV}$. What is the minimum proton energy to produce this reaction? What is the electron energy at this threshold proton energy? The proton mass is 938 MeV/c$^2$ and the electron mass is 0.5 MeV/c$^2$. 
A Michaelson interferometer has two arms, the first of length $L$ and the second of length $L+d$. The interferometer is illuminated by a polychromatic source of light with a frequency distribution given by

$$I(\nu) = \left( \frac{I_0}{\sqrt{2\pi\sigma^2}} \right) \exp\left[ -\left( \nu - \nu_0 \right)^2 / 2\sigma^2 \right]$$

a) Describe qualitatively how you expect the interference pattern for the case with polydisperse light to differ from the case where the interferometer is illuminated with a single frequency of light.

b) Calculate the intensity of light observed as a function of the arm offset $d$. Does this result confirm your prediction from part a?

You may want to use the following integral.

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2}} \cos^2(kx) \, dx = \sqrt{\frac{\pi\sigma^2}{2}} \left[ 1 + \cos(2ky)e^{-2ky^2} \right]$$