Solve 3 out of 4 problems. (40 points each. Total of 120 points)

1. [40 points] *A conducting sphere and a point charge*

   The field of a conducting sphere in the presence of a point charge can be described by
   \[ V(r) = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{|r + d|} + \frac{q'}{|r + d'|} \right) \]
   where \( q \) is the point charge and \( q' \) is an image charge inside the sphere representing the surface charge of the conducting sphere.

   (a) Determine \( q' \) and \( d' \) if the sphere is grounded assuming that the charge and image charge (located \( d \) and \( d' \), respectively) are on the z-axis. [10 points]

   (b) We now want to describe the conducting grounded sphere in a constant electric field. We can use the results of from (a) by considering that two equal and opposite charges at \( \pm d\hat{z} \) produce a constant electric field in the \( z \) direction \( \vec{E} = E\hat{z} \) if the distance between the charges \( 2d \) is significantly larger than the radius of the sphere \( R \). What is the potential in terms of the real and image charges. [10 points]

   (c) Expand the potential to lowest order in \( r/d \) in the limit that \( r/d \ll 1 \) and \( r/R \ll 1 \). Express the potential in terms of the applied electric field \( E_z \). [10 points]

   (d) Write the term due to the image charges in terms of the potential of an electric dipole. [10 points]

2. [40 points] *Partially filled coaxial conducting cylinders*

   The space between two coaxial conducting cylinders of length \( L \) is half-filled with a dielectric having relative dielectric constant \( \varepsilon_r \). The cylinders have radius \( r_1 \) and \( r_2 \), as shown in fig, and are connected to a \( V_1 \) battery.

   (a) Find the fields \( \vec{E} \) and \( \vec{D} \) in the air and in the dielectric in the space \( r_1 < r < r_2 \). [20 points]

   (b) Find the surface charge induced on the inner conductor at points adjacent to the air, and at points adjacent to the dielectric. [10 points]

   (c) Find the total charge on the inner conductor, and the capacitance. [10 points]
3. **[40 points] Non-magnetic metal disk under time-dependent external magnetic field.**

A non-magnetic metal disk of radius $a$, thickness $\ell$, and conductivity $\sigma$ is located parallel to the $xy$ plane, and centered at the origin. There is a slowly varying but time-dependent external uniform magnetic field $\vec{B} = B_0 \cos(\omega t) \hat{z}$.

(a) Find the induced current density $\vec{J}$ in the disk. [20 points]
(b) A very small wire loop, of radius $b$, and with resistance $R$, is located parallel to the disk, and centered above it at the point $(x, y, z) = (0, 0, h)$, with $h \gg a, b, \ell$. Find the current induced in the wire loop due to the magnetic field of the disk. [20 points]

4. **[40 points] Electron gun (Relativistic electron under electric field)**

Consider the electron gun in the figure below. The electrons travel from the planar cathode to the planar anode a distance $L$ away, and some of the beam is allowed to pass through the hole of radius $a \ll L$ into the field-free region on the right. In the vicinity of the hole, the field has radial components that deflect the electrons away from the axis.

(a) Use Gauss’s law to develop an approximation for the radial components of the field near the axis in terms of the longitudinal field $E_z(z)$ and its derivatives on the axis. [20 points]

(b) Assume that in the vicinity of the hole, the electrons follow nearly straight-line trajectories at constant velocity.
   i. Compute the change in the radial momentum as the electrons go through the hole to find the deflection of the electrons. Use $\beta c$ as the final relativistic velocity. [10 points]
   ii. Show that the electrons near the axis are defocused with a focal length
   $$ f = \frac{2\beta^2 \gamma}{\gamma - 1} L \approx 4L $$
   in the nonrelativistic limit, where $\beta c$ is the final relativistic velocity and $\gamma mc^2$ the final energy of the electrons, where $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$. As the figure shows, the focal length is for the diverging electrons! Note that the focal length is independent of the electron charge, the electron mass, the radius of the hole and (in the nonrelativistic limit) the electron final energy. [10 points]