Problem 1: An infinite flat sheet of uniform charge density [40 points]

An infinite flat sheet of uniform charge density $\sigma$, located in the x, y plane, oscillates along the x-axis. The velocity of the charges is given by $\vec{v} = v_0 \hat{x} \cos(\omega_0 t)$.

a. Find the electric and magnetic fields everywhere. [30 points]
b. What is the average power radiated per unit area of the sheet? [10 points]

Problem 2: Meissner effect [40 points]

The magnetic scalar potential outside a spinning shell (radius $R$, angular velocity $\omega$, total charge $q$) of charge is:

$$\Phi_{\text{out}} = -\frac{\mu_0 \omega q R^2}{12 \pi r^2} P_1(\cos \theta)$$

a. Show that the magnetic field outside a spinning spherical shell of charge is

$$\vec{B}(\vec{r}) = -\nabla \Phi_M(\vec{r}) = \frac{\mu_0}{4\pi} \left( \frac{3\vec{m} \cdot \vec{r}}{r^3} \vec{r} \cdot \vec{r} r^3 - \frac{\vec{m}}{r^3} \right)$$

where the magnetic dipole moment of the spinning sphere is

$$\vec{m} = \frac{1}{3} q R^2 \omega.$$ [20 points]

b. Below the critical magnetic field $H_c$, type I superconductors exhibit the Meissner effect, in which all magnetic fields are excluded from the volume of the superconductor. Consider a superconducting sphere of radius $a$ placed in an otherwise uniform magnetic induction $B_0$. Since the magnetic field is excluded from the sphere, the magnetic induction $B$ is tangent to the sphere at its surface as shown in Figure below.

i. Show that the total magnetic induction outside the sphere is

$$\vec{B}(\vec{r}) = \vec{B}_0 + \frac{\mu_0}{4\pi} \left( \frac{3\vec{m} \cdot \vec{r}}{r^3} \vec{r} \cdot \vec{r} r^3 - \frac{\vec{m}}{r^3} \right)$$

where the induced dipole moment is

$$\vec{m} = -\frac{2\pi a^3}{\mu_0} \vec{B}_0.$$ [10 points]

ii. Calculate the energy required to place the sphere in the magnetic field. [hint: the magnetic field induction exerts a pressure on the surface of the sphere. Compute $\delta W$ done on the field first when the radius of the sphere expands from $r$ to $r + \delta r$.] [10 points]
Problem 3: Space charge limited current in a diode [40 points]

Consider the space charge-limited flow of current in a planar vacuum diode. We assume that the cathode represents an unlimited source of electrons at zero velocity that are accelerated by the electric field toward the anode. The electrons in the anode-cathode gap form a space charge density

\[ \rho = \frac{J}{v} \]

where \( J \) is the current density and \( v \) the electron velocity, that modifies the field and repels the electrons leaving the cathode. The current becomes limited when the electric field at the surface of the cathode vanishes.

a. Express the Poisson’s equation for this problem for non-relativistic motion of electrons. [20 points]

b. For nonrelativistic motion of the electrons, calculate the space charge-limited current in the diode. \( V \) is the potential of the anode relative to the cathode. The quantity

\[ K = \frac{4\varepsilon_0 A}{9d^2} \sqrt{\frac{2e}{m}} \]

where \( A \) is the area of the cathode, \( d \) the anode-cathode separation, \( e = |q| \) the magnitude of the electron charge, and \( m \) the electron mass, is called the perveance of the diode. [20 points]

Problem 4: Paradox? [40 points]

Consider a point charge \( q \) moving at constant relativistic velocity \( v \) along the \( z \)-axis.

1. Write down the expression for the particle’s electric field in all direction perpendicular to its velocity. How does the field depend on \( \gamma \) the Lorentz factor? [10 points]

2. Compute the electric flux \( \Phi_E \) from the particle using Gauss’s law. [Hint: the integral

\[ \int_0^\pi dz \frac{\sin z}{\left[1 - \beta^2 \sin^2 z \right]^{3/2}} = 2\gamma^2 \]

might turn out to be useful…]. [20 points]

3. Compare your answer from (1) and (2) and try to explain the apparent “discrepancy” in the \( \gamma \)-dependence. [10 points]