You may solve ALL FOUR problems if you choose. The points of the best three problems will be counted towards your final score of this part of the examination. (40 points each. Total of 120 points)

Do not just quote a result. Show your work clearly step by step.

1. [40 points] A static electric field with spherical symmetry is described by

\[ E(r) = c r e^{-r/R} \hat{r}, \]

where \(c\) and \(R\) are constants.

(a) Determine the charge density, \(\rho(r)\). [8 points]
(b) Find the total charge of the system. [8 points]
(c) Find the static electric potential \(V(r)\). [8 points]
(d) Find the total energy stored in the electric field. [8 points]
(e) A small test charge \(+q\) is released at rest at the radial location \(r = R\). What is the kinetic energy when it reaches a point far away? [8 points]

2. [40 points] A toy consists of identical donut-shaped permanent magnets (magnetization parallel to the axis), which slide on a vertical rod without friction (Fig. 1). Treat the magnets as dipoles with mass \(m_d\) and dipole moment \(\vec{m}_d\). (a) and (b) are questions on a magnetic dipole in general, and (c) and (d) are specific to this magnet toy, using the results of the general dipole.

(a) In general, what is the vector potential of a (pure) magnetic dipole \(\vec{A}_{\text{dip}}(r)\) if the magnetic dipole \(\vec{m}\) is at the origin and points in the z-direction (Fig.2)? Express in the spherical coordinates. [6 points]
(b) Then, still in general, what is the magnetic field of a (pure) dipole \(\vec{B}_{\text{dip}}(r)\)? Express in the spherical coordinates. [4 points]
Now let’s consider the toy mentioned above.

(c) If one puts two back-to-back magnets on the rod, the upper one will “float” – the magnetic force upward balancing the gravitational force downward (Fig. 1a). At what height \( z \) does it float? [15 points]

(d) Now one adds a third donut-shaped magnet (parallel to the bottom one) as in Fig. 1b. Show that the ratio of the two heights \( \alpha \equiv \frac{x}{y} \) is expressed as

\[
2 = \left(\frac{1}{\alpha^2}\right) + \left(\frac{1}{(\alpha + 1)^2}\right).
\]

\( \alpha = 0.850115 \) (numerically calculated). [15 points]

3. [40 points] A rectangular loop (with sides \( a \) and \( b \)) is located the magnetic field produced by an infinitely long current-carrying straight wire; see Fig. 3.

(a) Find the induced voltage in the loop if it is fixed in space while the current in the long wire is time dependent: \( I(t) = I_0 \cos(\omega t) \). [15 points]

(b) Find the magnitude and direction of the induced voltage as a function of \( r \) when the current is constant in time \( [I(t) = I_0] \), while the loop moves towards the wire with velocity \( u \). [15 points]

(c) Find the induced voltage in the loop when the current is \( I(t) = I_0 \cos(\omega t) \) and loop moves towards the wire with velocity \( u \). [10 points]

![Fig. 3. Diagram for Problem 3.](image)

4. [40 points] An electromagnetic wave propagates in a long coaxial cable consisting of an inner cylinder of radius \( a \) and an outer cylinder of radius \( b \). Suppose that the magnetic field for \( a < r < b \) is given in cylindrical coordinates by

\[
\vec{B} = B_0 \left(\frac{r}{a}\right)^n \cos(kz - \omega t) \hat{\phi}
\]  

(1)

where \( B_0 \), \( \omega \), and \( k \) are positive constants and \( n \) is a rational number. The region \( a < r < b \) is vacuum.

(a) Solve for the constant \( n \). [8 points]

(b) Find the electric field in the region \( a < r < b \). [8 points]

(c) Find the relation between \( k \) and \( \omega \). [6 points]

(d) Find the charge density and the current density on the inner cylinder. [12 points]

(e) Find the total time-averaged power transmitted by the cable. [6 points]
You may solve ALL FOUR problems if you choose. The points of the best three problems will be counted towards your final score of this part of the examination. (40 points each. Total of 120 points)

Do not just quote a result. Show your work clearly step by step.

1. [40 points] A spherical shell, of radius R, carrying a uniform surface charge $\sigma$, is set spinning at angular velocity $\omega$.
   
   (a) Find the vector potential it produces at point $r$ (both $r \leq R$ and $r \geq R$). [20 points]
   
   (b) Find the magnetic field inside this spherical shell. [10 points]
   
   (c) Find the magnetic field outside this spherical shell. [10 points]

2. [40 points] A sphere of radius $a$ has a bound charge $Q$ distributed uniformly over its surface. The sphere is surrounded by a uniform fluid dielectric medium with fixed dielectric constant $\varepsilon$, as shown in Fig. 1. The fluid also contains a free charge density given by
   
   $\rho(r) = -k V(r)$,
   
   where $k$ is a constant and $V(r)$ is the electric potential at $r$ relative to infinity.

   (a) Find the potential everywhere, letting $V = 0$ at $r \to \infty$. [20 points]
   
   (b) Find the pressure as a function of $r$ in the dielectric. [20 points]

3. [40 points] Consider a very long solenoid with radius $R$, $n$ turns per unit length, and current $I$. Coaxial with the solenoid are two long cylindrical shells of length $l$. One is inside the solenoid at radius $a$, carries a charge $+Q$, uniformly distributed over its surface. The other is outside the solenoid at radius $b$, carries charge $-Q$. $l >> b$. When the current in the solenoid is gradually reduced, the cylinders begin to rotate.

   (a) Sketch the set up of the above solenoid-two cylindrical shell system with relevant physical parameters, charges, and the electrical current in the cylindrical coordinate system ($\hat{s}, \hat{\phi}, \hat{z}$). [4 points]

   (b) Before the current $I$ was switched off, find the electric field in the region between the cylinders ($a < s < b$) $E$, a magnetic field inside the solenoid $B$, and the total angular momentum in the fields $L_{em}$. [16 points]

   (c) When the current $I$ was switched off, find induced circumferential electric fields $E$ inside ($s > R$) and outside ($s < R$) of the solenoid, the torque $N_b$ and an angular momentum on the outer cylinder $L_b$, and the torque $N_a$ and an angular momentum on the inner cylinder $L_a$. What is the total angular momentum $L_{em}$? [16 points]

   (d) Summarizing the above, is the total angular momentum (fields plus matter) $L_{em}$ conserved before and after the current $I$ was switched off? Yes or No. [4 points]
4. [40 points] An electromagnetic wave moving in vacuum in the $+\hat{z}$ direction with angular frequency $\omega$ is normally incident at the plane of interface ($z = 0$) with an Ohmic conducting material. The vacuum occupies the region $z < 0$, while the conductor has conductivity $\sigma$ and occupies the region $z > 0$. The conductor has the same electric permittivity and magnetic permeability as vacuum (so $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$). The electric field amplitude of the wave just inside the conductor is $E_0$.

(a) [25 points] In the limit of high conductivity, $\sigma \gg \omega \varepsilon_0$, calculate how deeply the electric field penetrates into the conductor before decreasing to $1/e$ of its amplitude at the surface.

(b) [15 points] Find the time-averaged power absorbed per unit area of the conductor.
1. **Charge-limited emission in an electron gun:** We consider an electron gun diagrammed in the figure below. An electrostatic field $E_a$ is applied between the cathode and anode planes. A thin sheet of electron [with infinitely small length, surface charge density $-\sigma$ (in our convention $\sigma > 0$), and total charge $Q$] is emitted from the cathode plane located at $z=0$. We consider the sheet to be an infinite plane and assume it has a vanishing velocity at the emission time ($t=0$).

   (a) Compute the electric field $E_b$ generated by the electron sheet at any axial distance $z$ (both magnitude and direction) when the sheet is at $z=d$. Assuming the applied field $E_a$ is small enough to ensure the dynamics of the electron sheet remains non-relativistic, give the kinetic energy of the electron sheet as function of $z$? [12 points]

   (b) We now consider a second *identical* thin sheet emitted at a time $\tau$ after the first electron sheet (discussed in previous questions).

   i. What is the total force on this second sheet and how does it affects its motion. Especially compare the kinetic energy of the second sheet with respect to the first one at the time each reaches the anode at $z=L$?[7 points]

   ii. Give the axial separation between the two sheets when the first sheet is at $z=d$ in the limit when $\tau=0$. [6 points]

   iii. Using your answer from (a) and (b), explain how does the first sheet affects the dynamics of the second sheet and especially consider the case when the field from the first sheet $E_b$ become comparable or larger than the applied field $E_a$ [7 points]

   iv. Consider that the setup is used to emit a *continuous* electron beam. Based on the previous analysis what can you conclude on the maximum possible surface charge density of this beam. What would happen if one tries to produce higher surface charge densities? [8 points]
2. [40 points] A parallel-plate capacitor is made of circular plates as shown in Fig. 1. The voltage across the plates (supplied by long resistanceless lead wires) has the time dependence $V = V_0 \cos \omega t$. Assume $d << a << c/\omega$, so that fringing of the electric field and retardation may be ignored (Region (I): between two plates, Region (II): right above the upper plate. These two regions are sufficiently distant from the edges of the plates.).

(a) Use Maxwell’s equation and symmetry arguments to determine the electric and magnetic fields in region I as functions of time. [20 points]

(b) What current flows in the lead wires and what is the current density in the plates as a function of time? [20 points]


(a) Write down the integral and differential forms of Maxwell’s equations. [10 points]

(b) Set the source terms in the differential forms to zero and from the resulting equations derive the wave equations for the electric and magnetic fields. [15 points]

(c) Assume the electric field solution to the wave equation is $\vec{E}(\vec{r}, t) = E_0 \hat{x} \cos(kz - \omega t + \delta)$ and hence show $c = 1/(\mu_0 \varepsilon_0)^{1/2}$. [15 points]

4. [40 points] A small source of electromagnetic radiation is located near the origin. The potentials for large $r$ are given in spherical coordinates by:

$$V(\vec{r}, t) = -V_0 \cos(\omega(t - r/c)) \frac{\sin \theta}{r}$$

$$\vec{A} (\vec{r}, t) = A_0 \cos(\omega(t - r/c)) \frac{\sin \theta}{r} (\hat{\phi} + \hat{r}),$$

where $V_0$ and $A_0$ are constants.

(a) Find the electric and magnetic fields for large $r$ (consistently neglecting contributions that fall off faster than $1/r$). [13 points]

(b) Solve for $V_0$ in terms of other quantities, and use the result to simplify your answers for the fields. [13 points]

(c) Is this electric dipole radiation, magnetic dipole radiation, both, or neither? [5 points]

(d) A gauge transformation is performed, so that $V = 0$ in the new gauge. What is $\vec{A}$ in the new gauge? Again, neglect contributions that fall off faster than $1/r$. [9 points]
NIU Ph.D. Candidacy Examination Spring 2015 (2/20/2016)

Electricity and Magnetism

Solve 3 out of 4 problems. (40 points each. Total of 120 points)

Do not just quote a result, show your work clearly step by step.

1. [40 points] As in Fig. 1, you are given the not-so-parallel plate capacitor.
   
   (a) Neglecting edge effects, when a voltage difference $V$ is placed across the two conductors, find the potential everywhere between the plates. [20 points]
   
   (b) In case of this wedge filled with a medium of dielectric constant $\varepsilon$, calculate the capacitance of the system in terms of the constants given. [20 points]

   ![Fig. 1](image)

2. [40 points] Consider the Earth as a sphere with its mean radius of $R = 6371$ kmeters, and magnetically as a dipole situated at its center, with dipole moment magnitude $m = |\vec{m}| = 7.9 \times 10^{22}$ Ameters$^2$. [*In this problem, the units of meters are spelt out in order to distinguish it from the symbol $m$ for the magnetic moment.]

   (a) The dipolar magnetic fields have the following components:

   \[
   B_r = \frac{\mu_0 \cos \theta}{2\pi} \frac{m}{r^2}, \quad (1)
   \]

   \[
   B_\theta = \frac{\mu_0 \sin \theta}{4\pi} \frac{m}{r^2}, \quad (2)
   \]

   \[
   B_\phi = 0, \quad (3)
   \]

   in spherical coordinates $(r,\theta,\phi)$. What is the value of $k$? Justify your answer with mathematical details. [15 points]

   (b) The Maxwell tensor can be represented in component form as follows:

   \[
   T_{ij} = E_i D_j + H_i B_j - \frac{1}{2} \delta_{ij} U_V \quad (4)
   \]

   What is the rank of the tensor? Define every symbol that appears in (4), and give the explicit definition of $U_V$ [15 points]

   (c) Derive the expression of the mechanical stress in the radial direction at the Earth’s magnetic North pole, and then estimate the order of magnitude in N/meters$^2$. [10 points]
3. [40 points] Consider a small circular conducting wire ring of radius \( a \) and resistance \( R \), which lies in the \( xy \) plane with its center at the origin. In the following, neglect the self-inductance of the ring; this turns out to be a good approximation in the limit of large resistance \( R \).

Suppose the ring is placed in a uniform but time-dependent magnetic field \( \vec{B} = B_0 \cos(\omega t) \hat{z} \).

(a) What is the current induced in the ring as a function of \( t \)? How much total energy is lost to heating the ring, during a very long time \( T \)? [15 points]

(b) Assuming that \( \omega \) is not too large, find the total energy lost by the ring to electromagnetic radiation, in a very long time \( T \). [20 points]

(c) Find the value of the resistance \( R \) such that the energy lost to radiation in part (b) is equal to the energy lost to heat in part (a). [5 points]

 Possibly useful information: far from an oscillating magnetic dipole moment, \( m_0 \cos(\omega t) \hat{z} \), the magnetic field expressed in spherical coordinates is given by

\[
\vec{B} = -\frac{\mu_0 m_0 \omega^2 \sin \theta}{4\pi c^2} \frac{\cos[\omega (t - r/c)]}{r} \hat{\theta} \tag{1}
\]

4. [40 points] Suppose velocity \( \vec{v} \) and acceleration \( \vec{a} \) of a moving charged particle are collinear (at time of \( t_r \), as in the regarded time), as for example, in straight-line motion. Here, velocity \( \vec{v} \) is along the \( z \) axis as in Fig. 2.

(a) Show that the angular distribution of the radiation is expressed as the power \( P \) radiated by the particle into the solid angle \( \Omega \),

\[
\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}
\]

where \( \beta \equiv v/c \), \( q \) is the charge of the particle. [20 points]

(b) Find the angle \( \theta_{\text{max}} \) at which the maximum radiation is emitted. [10 points]

(c) Find the total power emitted. [10 points]
Solve 3 out of 4 problems. (40 points each. Total of 120 points)

Do not just quote a result, show your work clearly step by step.

1. [40 points]
   (a) Find the force on a square loop placed as shown in Fig 1, near an infinite straight wire. Both the loop and the wire carry a steady current \( I \). [20 points]
   (b) Suppose the current \( I(t) \) in the long straight wire in Fig 1 is changing slowly according to \( I(t) = I_0 \cos(\omega t) \). Find the current induced in the square loop as a function of time, if it has resistance \( R \). [20 points]

2. [40 points] A capacitor is made of two plane parallel plates of width \( a \) and length \( b \) separated by a distance \( d \) \((d \ll a, b)\), as in Fig. 2. The capacitor has a dielectric slab of relative dielectric constant \( K \) between the two plates.
   (a) The capacitance is connected to a battery of emf \( V \). The dielectric slab is partially pulled out of the plates such that only a length \( x \) remains between the plates. Calculate the force on the dielectric slab which tends to pull it back into the plates. [20 points]
   (b) With the dielectric slab fully inside, the capacitor plates are charged to a potential difference \( V \) and the battery is disconnected. Again, the dielectric slab is pulled out such that only a length \( x \) remains inside the plates. Calculate the force on the dielectric slab which tends to pull it back into the plates. [20 points]
3. [40 points] A soap film of thickness $a$ and permittivity $\varepsilon$ is suspended in empty space (Fig. 3). The permittivity of the soap film is very large compared to that of vacuum, $\varepsilon >> \varepsilon_0$. Two charged ions, each of charge $+Q$, are located a distance $R$ apart, in the midplane of the film. Find the force between the ions in the three limiting cases:

(a) if $R \ll a$ [13 points]

(b) if $a \ll R \ll ea$ [14 points]

(c) if $ea \ll R$. [13 points]

![Fig. 3]

4. [40 points] Consider the following idealized situation with an infinitely long, thin, conducting wire along the $z$ axis. For $t < 0$ it is current-free, but at $t = 0$ a constant current density $\vec{J}$ is applied simultaneously over the entire length of the wire. Consequently, the wire carries the current

$$\vec{J} = \begin{cases} 0, & t < 0; \\ \frac{\varepsilon}{\mu_0} \frac{\partial \vec{E}}{\partial t}, & t \geq 0. \end{cases}$$

(1)

It is assumed that the conductor can be kept uncharged, i.e. $\rho = 0$.

(a) Determine scalar and vector potentials induced everywhere in space, $\phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ as functions of time. [10 points]

(b) Determine the magnetic and electric fields induced everywhere in space, $\vec{B}(\vec{x}, t)$ and $\vec{E}(\vec{x}, t)$ as functions of time. [20 points]

(c) Calculate the total power radiated per unit wire length. Comment on the unphysical behavior at $t = 0$ and its explanation for realistic systems. [10 points]
NIU Ph.D. candidacy examination Spring 2015 (2/21/2015)

Electricity and Magnetism

Solve 3 out of 4 problems. (40 points each. Total of 120 points)

Do not just quote a result, show your work clearly step by step.

1. [40 points]
   (a) For a uniformly charged sphere (radius \( R \), charge density \( \rho \)),
   i. Find the electric field everywhere. [15 points]
   ii. Find the energy stored in this configuration. [15 points]

   (b) Two spheres, each of radius \( R \) and carrying uniform charge densities \( +\rho \) and \( -\rho \),
   respectively, are placed so that they partially overlap as in the figure below. The
   vector \( \vec{d} \) is the vector from the positive center to the negative center. Find the value
   of the field in the region of overlap using \( \vec{d} \). [10 points]

2. [40 points] A toroidal coil has a rectangular cross section, with inner radius \( a \), outer radius \( a + w \), and height \( h \). It carries a total of \( N \) tightly wound turns, and the current is increasing at a constant rate (\( \frac{dI}{dt} = k \)).
   (a) In the quasistatic approximation, find the direction and the magnitude of the magnetic
       field \( \vec{B} \) everywhere (for points inside and outside of the toroidal coil). [6 points]
   (b) Calculate the flux of \( \vec{B} \) through the cross section the toroid. [14 points]
   (c) Assume \( w \) and \( h \) are both much less than \( a \). Find the induced electric field \( \vec{E} \) at a point \( z \)
       above the center of the toroid. [20 points]
3. [40 points] An electric dipole moment $\vec{p}(t) = p_0 \cos(\omega t) \hat{z}$ oscillating along the z axis generates radiating electric and magnetic fields. Far away from the dipole, the scalar and vector potentials due to this dipole are:

$$V = -\frac{p_0 \omega}{4\pi \epsilon_0 c} \left( \frac{\cos \theta}{r} \right) \sin \left[ \omega \left( t - \frac{r}{c} \right) \right]$$

$$\vec{A} = \frac{X}{r} \sin \left[ \omega \left( t - \frac{r}{c} \right) \right] \hat{z}$$

in SI units where $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light, and $(r, \theta, \phi)$ are the usual spherical coordinates, and $X$ is a constant.

(a) Use Maxwell’s equations to derive the constant $X$ in terms of the other quantities. (Do not just quote a result.) [15 points]

(b) Derive, from the results above, the total time-averaged power emitted from this dipole. (Do not just quote a result.) [15 points]

(c) What is the ratio of the time-averaged power per unit area received by two detectors, one at $(x, y, z) = (D, 0, D)$ and the other at $(D, 0, 0)$, where $D$ is a very large distance? [10 points]

4. [40 points]

In a certain reference frame $S$, a static uniform electric field $\vec{E}_0$ is parallel to the x-axis, and a static uniform magnetic field $\vec{B}_0$ with magnitude $cB_0 = 2E_0$ lies in the xy-plane, making an angle $\theta = 30^\circ (\neq 0)$ with respect to the x-axis as shown in the figure below. Here, $c$ is the speed of light in vacuum. Determine the scaled relative velocity (relativistic $\beta$) $\beta = v/c$ of another reference frame $S'$ moving in the z-direction relative to $S$, $\vec{v} = v \hat{z}$, in which the electric and magnetic fields are parallel.

The reference frame’s $S'$ velocity relative to $S$ is $\vec{v} = v \hat{z}$
Solve 3 out of 4 problems. (40 points each. Total of 120 points)

1. [40 points] \textit{A hole in a large conducting slab}

Consider a conducting slab of material of uniform conductivity $\sigma$, extending infinitely in the x and y directions and from $-b < z < b$. A cylindrical hole is made through the slab with radius $a$ and axis along the z axis. Far from the hole, the slab carries a uniform current density $J = J_0 \hat{x}$. In the following, assume $a \ll b$ and consider only the region of small $|z| \ll b$, so that the z dependence can be neglected.

(a) Determine the electric potential distributions in the slab and in the hole, respectively. [10 x 2 = 20 points]
(b) Find the electric field in the hole. [10 points]
(c) The space of the hole is filled with a material of conductivity $\sigma'$. Determine the current density in this space. [10 points]

2. [40 points] \textit{A linear dielectric sphere}

A linear dielectric sphere of radius $a$ and dielectric constant $\kappa$ carries a uniform charge density $\rho$, surrounded by vacuum.

(a) Find $E$ and $D$ inside and outside the sphere. [15 + 10 points]
(b) Find the energy W of the system. [15 points]

3. [40 points] \textit{The magnetic field by an infinitely long cylinder with a “frozen-in” magnetization}

An infinitely long cylinder (radius R) carries a “frozen-in” magnetization parallel to the axis, $\mathbf{M} = ks \hat{z}$, where $k$ is a constant and $s$ is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside of the cylinder by two different methods (a) and (b):

(a) Locate all the bound currents, and calculate the field $\mathbf{B}$ they produce. [20 points]
(b) Use Ampere’s law to find $\mathbf{H}$, and then get $\mathbf{B}$. [20 points]

Note: Of course, you should get the same final answers obtained by the above two methods. Therefore, expressing the derivations of the answers are the most important point for this problem.

4. [40 points] \textit{Charged cylindrical tube rotating with increasing angular velocity}

A long thin-walled cylindrical tube of radius R carries a uniform surface charge density $\sigma$, and is rotating about its axis of symmetry with a constant angular acceleration $\alpha$, so that its angular velocity at time $t$ is $\omega = \alpha t$.

(a) Find all of the components of the electric and magnetic fields both inside and outside of the cylinder. [26 points]
(b) Consider a length $\ell$ of the tube. Find the total flux of Poynting’s vector into the tube through its outside surface of radius slightly larger than R, and show that it equals the time rate of change of the stored electromagnetic energy within the volume. [Hint: the result should be non-zero.] [14 points]
1. [40 points] A conducting sphere and a point charge

The field of a conducting sphere in the presence of a point charge can be described by

\[ V(r) = \frac{1}{4\pi \varepsilon_0} \left( \frac{q}{|r + d|} + \frac{q'}{|r + d'|} \right) \]

where \( q \) is the point charge and \( q' \) is an image charge inside the sphere representing the surface charge of the conducting sphere.

(a) Determine \( q' \) and \( d' \) if the sphere is grounded assuming that the charge and image charge (located \( d \) and \( d' \), respectively) are on the \( z \)-axis. [10 points]

(b) We now want to describe the conducting grounded sphere in a constant electric field. We can use the results of from (a) by considering that two equal and opposite charges at \( \pm d\hat{z} \) produce a constant electric field in the \( z \) direction \( \vec{E} = E\hat{z} \) if the distance between the charges \( 2d \) is significantly larger than the radius of the sphere \( R \). What is the potential in terms of the real and image charges. [10 points]

(c) Expand the potential to lowest order in \( r/d \) in the limit that \( r/d \ll 1 \) and \( r/R \ll 1 \). Express the potential in terms of the applied electric field \( E_z \). [10 points]

(d) Write the term due to the image charges in terms of the potential of an electric dipole. [10 points]

2. [40 points] Partially filled coaxial conducting cylinders

The space between two coaxial conducting cylinders of length \( L \) is half-filled with a dielectric having relative dielectric constant \( \varepsilon_r \). The cylinders have radius \( r_1 \) and \( r_2 \), as shown in fig, and are connected to a \( V_1 \) battery.

a) Find the fields \( \vec{E} \) and \( \vec{D} \) in the air and in the dielectric in the space \( r_1 < r < r_2 \). [20 points]

b) Find the surface charge induced on the inner conductor at points adjacent to the air, and at points adjacent to the dielectric. [10 points]

c) Find the total charge on the inner conductor, and the capacitance. [10 points]
3. [40 points] **Non-magnetic metal disk under time-dependent external magnetic field.**

A non-magnetic metal disk of radius $a$, thickness $\ell$, and conductivity $\sigma$ is located parallel to the $xy$ plane, and centered at the origin. There is a slowly varying but time-dependent external uniform magnetic field $\vec{B} = B_0 \cos(\omega t)\hat{z}$. 

(a) Find the induced current density $\vec{J}$ in the disk. [20 points]
(b) A very small wire loop, of radius $b$, and with resistance $R$, is located parallel to the disk, and centered above it at the point $(x, y, z) = (0, 0, h)$, with $h \gg a, b, \ell$. Find the current induced in the wire loop due to the magnetic field of the disk. [20 points]

4. [40 points] **Electron gun (Relativistic electron under electric field)**

Consider the electron gun in the figure below. The electrons travel from the planar cathode to the planar anode a distance $L$ away, and some of the beam is allowed to pass through the hole of radius $a \ll L$ into the field-free region on the right. In the vicinity of the hole, the field has radial components that deflect the electrons away from the axis.

(a) Use Gauss’s law to develop an approximation for the radial components of the field near the axis in terms of the longitudinal field $E_z(z)$ and its derivatives on the axis. [20 points]
(b) Assume that in the vicinity of the hole, the electrons follow nearly straight-line trajectories at constant velocity.
   i. Compute the change in the radial momentum as the electrons go through the hole to find the deflection of the electrons. Use $\beta c$ as the final relativistic velocity. [10 points]
   ii. Show that the electrons near the axis are defocused with a focal length $f = \frac{2\beta^2 \gamma}{\gamma - 1}L = 4L$ in the nonrelativistic limit, where $\beta c$ is the final relativistic velocity and $\gamma mc^2$ the final energy of the electrons, where $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$. As the figure shows, the focal length is for the diverging electrons! Note that the focal length is independent of the electron charge, the electron mass, the radius of the hole and (in the nonrelativistic limit) the electron final energy. [10 points]
Solve 3 out of 4 problems. Total of 120 points.

1. **[40 points] Linear dielectric material.** The space between the plates of a parallel-plate capacitor (Fig. 1) is filled with two slabs of linear dielectric material. Each slab has thickness $a$, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of $1.5$. The free charge density on the top plate is $\sigma$ and on the bottom plate $-\sigma$.
   a. Find the electric displacement $D$ in each slab.
   b. Find the electric field $E$ in each slab.
   c. Find the polarization $P$ in each slab.
   d. Find the potential difference between the plates.
   e. Find the location and amount of all bound charge.
   f. From all of the above charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

2. **[40 points] A magnetic dipole $\mathbf{m}$ is imbedded** at the center of a sphere (radius $R$) of linear magnetic material (permeability $\mu$). There is a bound dipole at the center. So the net dipole moment at the center $\mathbf{m}_{\text{center}}$ produces a magnetic filed $\mathbf{B}_{\text{center}}$. Here magnetic susceptibility is given as $\chi_m$.
   a. Express $\mathbf{m}_{\text{center}}$ with $\mathbf{m}$. [5 points]
   b. Then, show that the magnetic field of a dipole can be written in coordinate-free form
      \[
      \mathbf{B}_{\text{center, dipole}} (\mathbf{r}) = \frac{\mu}{4\pi} \frac{1}{r^2} \left[ 3 (\mathbf{\hat{r}} \cdot \mathbf{m}) \mathbf{\hat{r}} - \mathbf{m} \right].
      \]  
      [10 points]
   c. Assuming $\mathbf{B}_{\text{surface, current}}$ is proportional to $\mathbf{m}$, find the magnetization $\mathbf{M}$. [7 points]
   d. Find the bound surface current $\mathbf{K}_b$. [8 points]
   e. Find the magnetic field due to the bound surface current $\mathbf{B}_{\text{surface, current}}$. [5 points]
   f. Finally express the magnetic field inside the sphere $\mathbf{B} (0 < r \leq R)$. [5 points]
3. [40 points] Electromagnetic fields candidates. Let $E_0$, $B_0$, $k$, and $\omega$ be four parameters and consider the following two vector fields:

Case (1): $\mathbf{E}(x,t) = E_0 \cos (kx - \omega t) \mathbf{\hat{z}}$ and $\mathbf{B}(x,t) = B_0 \cos (kx - \omega t) \mathbf{\hat{z}}$,

Case (2): $\mathbf{E}(x,t) = E_0 \cos (kx - \omega t) \mathbf{\hat{z}}$ and $\mathbf{B}(x,t) = B_0 \cos (kx - \omega t) \mathbf{\hat{y}}$.

The fields are defined over the entire space $[\text{coordinate } x = (x,y,z)]$ and for arbitrary time $t$. The ^ symbol represents unit vectors.

a. Can any of these fields represent an electromagnetic field? Consider both cases [Cases (1) and (2)] and for each case detailed your reasoning. Give the physical meaning of the four parameters. [10 points]

b. Assuming your answer to question (1) is positive for at least one of the cases,
   i. give the necessary condition(s) between the parameters $E_0$, $B_0$, $k$, and $\omega$. [10 points]
   ii. find the corresponding charge distribution and current source, $\rho(x,t)$ and $J(x,t)$, respectively. [10 points]

c. Consider the case $\omega = k/\sqrt{\varepsilon_0 \mu_0}$. What are the corresponding charge distribution and current source? For this case find the Poynting vector and its time-averaged value. [10 points]

4. [40 points] Resonant cylindrical-symmetric cavity for non-relativistic beams: We consider a resonant cylindrical-symmetric cavity operating on the TM$_{010}$ mode. The axial (accelerating) field on the cavity axis is taken to be $E_z(r = 0, z, t) = E_0 \sin (\omega t)$.[40 points]

a. We consider a “reference” charged particle with charge $q$, mass $m$ and initial energy $W_r$ entering the cavity and crossing its center at time $t = 0$. Assume that the particle velocity does not change within the cavity. What is its final energy? [7 points]

b. Consider a second identical particle with same initial energy as the reference particle but delayed by a time $\delta t$ small. What is its final energy $W$? [7 points]

c. The two particles are allowed to drift in free space over a length $L$ downstream of the accelerating cavity.
   i. Find the final arrival time at the end of the drift associated to each of the particles. [10 points]
   ii. Express the difference in final arrival times as a function of $\delta \gamma = \frac{W - W_r}{mc^2}$, $\gamma = \frac{W + W_r}{2mc^2}$ the mean Lorentz factor of the two-particle system, and $\delta t$. [10 points]
   iii. What is the condition between $\delta \gamma / \gamma$ and $\delta t$ that insures the two particles arrive at the same time (i.e. are bunched)? [6 points]
Solve 3 out of 4 problems. (40 points each. Total of 120 points)

1. [40 points] **Infinitely long uniform line charge.**
   A uniform line charge \( \lambda \) is placed on an infinite straight wire, a distance \( d \) above a grounded conducting plane. The wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy-plane.
   
   a) Find the potential in the region above the plane. [20 points]
   b) Find the charge density \( \sigma \) induced on the conducting plane. [20 points]

2. [40 points] **Wire with hole.**
   The figure shows the cross section of an infinitely long circular cylinder of radius 3a with an infinitely long cylindrical hole of radius a displaced so that the hole’s center is at a distance a from the center of the big cylinder. The solid part of the cylinder carries a current I, distributed uniformly over the cross section, and directed out of the plane of the paper.
   
   a) Find the magnetic field at all points in the plane P containing the axes of the cylinders (in the figure: the xz-plane with \( y = 0 \)). [20 points]
   b) Determine the magnetic field throughout the hole; it is of a particularly simple character. [20 points]
3. [40 points] **Rotating solid dielectric cylinder.**

A very long solid dielectric cylinder of radius $R$ with electric susceptibility $\chi_e$ is rotating at a constant angular velocity $\omega$ about its axis of symmetry, the $z$-axis. There is also a constant magnetic field in the $z$ direction: $\vec{B} = B\hat{z}$. Find:
   a) The electric dipole moment density (per unit volume) as a function of position within the cylinder, [18 points]
   b) The volume charge density as a function of position within the cylinder, [12 points]
   c) The surface charge density on the cylinder. [10 points]

4. [40 points] **Waveguide.**

Consider a rectangular waveguide, infinitely long in the $z$ direction, with transverse dimensions $L_x = L$ and $L_y = 2L$ as illustrated in the figure. The walls are a perfect conductor.
   a) What are the boundary conditions for the components of $B$ and $E$ at the walls? [8 points]
   b) Write the wave equation which describes the $E$ and $B$ fields of the lowest mode. (Hint: The lowest mode has the electric field in the $x$ direction only.) [8 points]
   c) For the lowest mode that can propagate, find the phase velocity and the group velocity. [14 points]
   d) The possible modes of propagation such waveguides separate naturally into two classes. What are these two classes and how do they differ physically? [10 points]
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Solve 3 out of 4 problems. (40 points each. Total of 120 points)

1. [40 points] **Point charges.**
   Three point charges \((q, -2q, q)\) are located in a straight line with separation \(a\) and with the middle charge \((-2q)\) at the origin of a grounded conducting spherical shell of radius \(b\), as indicated in the sketch below:

   (a) Write down the potential due to the three charges *in the absence of* the grounded sphere. [10 points]
   (b) Find the limiting form of the potential as \(a \to 0\), but the product \(qa^2 = Q\) remains finite. [15 points]
   (c) Write this latter answer in spherical coordinates. [15 points]

2. [40 points] **Electric dipole.**
   A perfect dipole \(p\) is located at a distance \(z\) above an infinite grounded conducting plane as shown below. The dipole makes an angle \(\theta\) with a perpendicular to the plane.

   a) Find the torque on \(p\). [25 points]
   b) If the dipole is free to rotate, in what orientation will it come to rest? [15 points]
3. [40 points] **Magnetic charges**

We assume the existence of magnetic charge \( \rho_m \) ("magnetic monopoles") related to the magnetic field \( \vec{B} \) by the local relation \( \nabla \cdot \vec{B} = \mu_0 \rho_m \), analogous to Gauss' law for electric charges.

a) Using the divergence theorem, obtain the magnetic field due to a point magnetic charge located at the origin \( r = 0 \). [10 points]

b) In the absence of magnetic charge, the curl of the electric field is given by Faraday's law in differential form. Show that this law is incompatible with a magnetic charge density that is a function of time. [10 points]

c) Assuming that magnetic charge is conserved, derive the *local* relation between the magnetic charge density \( \rho_m \) and the magnetic charge current density \( j_m \). Interpret your result. [10 points]

d) Modify Faraday's law to obtain a law consistent with the presence of a magnetic charge density, \( \rho_m = \rho_m(r, t) \). Demonstrate that the modified law is consistent with your solution to part (c). [10 points]

4. [40 points] **TEM mode propagated along a transmission line.**

A transmission line consisting of two concentric circular cylinders of metal with conductivity \( \sigma \) and skin depth \( \delta \), as shown, is filled with a uniform lossless dielectric \( (\mu, \varepsilon) \). A TEM mode is propagated along this line.

![Transmission line diagram]

a) Show that the time-averaged power flow along this line is

\[
P = \frac{\mu}{\varepsilon} \pi a^2 |H_0|^2 \ln \left( \frac{b}{a} \right)
\]

where \( H_0 \) is the peak value of the azimuthal magnetic field at the surface of the inner conductor. [10 points]

b) The transmitted power is attenuated along the line in the form of \( P(z) = P_0 e^{-2\gamma z} \). Express \( \gamma \) with conductivity \( \sigma \), skin depth \( \delta \), and dielectric constants of \( (\mu, \varepsilon) \). [10 points]

[Hint] The power loss per unit length of the waveguide is expressed as:

\[
-\frac{dP}{dz} = \frac{1}{2\sigma\delta} \oint_C n \times \vec{H}^2 dl = \frac{1}{2\sigma\delta} |H_0|^2 \oint_C \frac{a^2}{\rho^2} dl,
\]

where \( \rho \) is the radial cylindrical coordinate and \( C \) is a circular contour on the surface of the conductor.
c) The characteristic impedance of the line, $Z_0$, is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position $z$. Express the characteristic impedance $Z_0$ for this line. [10 points]

d) Express the inductance per unit length of the line, $L$, where $\mu_C$ is the permeability of the conductor. Take into account the correction to the inductance due to the penetration of the flux into the conductors by a distance of order $\delta$. [10 points]

[Hint] First express the energy per unit length stored in the magnetic field, $U_{vol}$ (inside the volume of the waveguide) and $U_{walls}$ (penetrated into the wall of the waveguide). Here the magnetic field penetrated into the conducting walls can be approximated to:

$$H(\zeta) = H \theta e^{-\zeta/\sigma} e^{i\zeta/\sigma},$$

where $\zeta$ is the distance into the conductor.
Solve 3 out of 4 problems.

1. [40 points] The electric field of a charged sheet.
   (a) Find the electric field at a height $z$ above the center of a square sheet (side $a$) carrying a uniform surface charge density $\sigma$. [20 points]
   (b) Find the electric field, keeping the leading non-zero term, when $a \to \infty$ (infinite plane). [10 points]
   (c) Find the electric field, keeping the leading non-zero term, when $z >> a$. [10 points]

2. [40 points] Consider a uniform thin shell of charge spinning about the $z$ axis with the angular velocity $\omega$, with the total charge $q$ and the radius of the sphere $R$.
   (a) Express its magnetic dipole moment, $\vec{m}$. [12 points]
   (b) Express the magnetic scalar potential (as a function of radius $r$) outside the spinning shell of charge using $\omega$, $q$, $R$. [14 points]
   (hint: The magnetic scalar potential can be represented by the expansion
   \[
   \Phi_m = \sum_{m=0}^{\infty} a_m r^m P_m(\cos \theta) \quad \text{(Outside the sphere)}
   \]
   \[
   \Phi_m = \sum_{m=1}^{\infty} b_m r^m P_m(\cos \theta) \quad \text{(Inside the sphere)}
   \]
   where $P_m(\zeta)$ is the Legendre polynomials, $a_m$ and $b_m$ are some coefficients.
   (c) Find the magnetic field outside the spinning shell of charge, in terms of the magnetic dipole moment, $\vec{m}$. [14 points]

3. [40 points] Induction of a toroidal coil.
   (a) Find the self-inductance of a toroidal coil with rectangular cross section (inner radius $a$, outer radius $b$, height $h$), which carries a total of $N$ turns. [20 points]
   (b) Calculate the energy stored in this toroidal coil. [20 points]

4. [40 points] An electromagnetic wave with angular frequency $\omega$ moves through a material that obeys Ohm’s Law with conductivity $\sigma$. The permittivity and permeability of the material are the same as that of vacuum.
   (a) Derive the separate second-order wave equations for the electric and magnetic fields $\vec{E}$ and $\vec{B}$. [12 points]
   (b) Find expressions for the electric and magnetic fields of a wave moving in the $\hat{z}$ direction and polarized in the $\hat{x}$ direction. [16 points]
   (c) Find the distance that the wave travels for which its intensity is decreased by a factor of 10. [12 points]
Solve 3 out of 4 problems.

1. [40 points] The time-averaged potential of a neutral atom is given by

   \[ \Phi = \frac{q}{4\pi\epsilon_0} \frac{e^{-\alpha r}}{r} (1 + \beta r), \]

   where \( q \) is the magnitude of the electronic charge, and \( \alpha^{-1} = a_0/2 \), \( a_0 \) being the Bohr radius.

   (a) Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically. [30 points]

   (b) Find the relationship between \( \alpha \) and \( \beta \) such that the charge distribution could describe a neutral atom. [10 points]

2. [40 points] The retarded vector potential in spherical coordinates \( (r, \theta, \phi) \) for a general oscillating magnetic dipole \( \vec{m} = m_0 \cos(\omega t) \hat{z} \) is

   \[ \vec{A} = \frac{\mu_0 m_0}{4\pi c} \left( \frac{\sin \theta}{r} \right) \cos \left[ \omega (t - r/c) \right] \hat{\phi}. \]

   Now consider a circular wire loop of radius \( a \) that is fed by a source of alternating current \( I = I_0 \cos(\omega t) \). The loop is centered at the origin and lies within the xy plane. Find the following in terms of \( a \), \( I_0 \), \( \omega \), constants of nature, and the spherical coordinates:

   (a) Calculate the leading behavior of the electric and magnetic fields in the large distance \( r \gg a, c/\omega \) limit. [20 points]

   (b) Describe the intensity pattern for the radiation as a function of angle, and find the total time-averaged radiated power. [20 points]

3. [40 points] A very long coaxial cylinder of length \( l \) is formed from an inner conductor of radius \( a \) and an outer conductor of radius \( b \). A solid dielectric plastic tube of permittivity \( \epsilon \) just fills the space between the conductors. Suppose that the charges on the inner and outer conducting cylinders are held constant at +Q and –Q, respectively.

   (a) Find the capacitance. [20 points]

   (b) If the plastic tube is withdrawn by a distance \( x \) from one end, find the magnitude and direction of the force on it. (Assume that the space left by the partially withdrawn plastic tube is air, with permittivity \( \epsilon_0 \).) [20 points]

4. [40 points] Consider the resonant cavity produced by closing off the two ends of a rectangular wave guide (cross sectional size \( a \) and \( b \)), at \( z = 0 \) and at \( z = d \), making a perfectly conducting empty box.

   (a) Find the resonant frequencies for TE mode. [20 points]

   (b) Find the associated electric field. [10 points]

   (c) Find the associated magnetic fields. [10 points]
Problem 1: [40 points]

An infinitely long strip (oriented along the y-axis) has width \( w \) and negligible thickness, see the figure. It carries a total current \( I \) uniformly distributed over the width.

a) Find expressions for the magnetic induction produced by this strip. [25 points]

b) Make sketches of \( B_z \) and \( B_x \) as function of \( x \) along traces at height \( z_0 \) above the strip for \( z_0/w \ll 1 \) and for \( z_0/w \gg 1 \) and discuss the results. [15 points]

Hints:

\[
\int \frac{x'}{(x')^2 + z_0^2} dx' = \frac{1}{2} \ln \left((x')^2 + z_0^2\right)
\]

\[
\int \frac{z_0}{(x')^2 + z_0^2} dx' = \arctan \left(\frac{x'}{z_0}\right)
\]

Problem 2: [40 points]

A surface charge density \( \sigma(x,y) = \sigma_0 \sin^2(\alpha x) \) is located on the xy-plane.

a) Find the electrostatic potential \( \phi(x,y,z) \) produced by this charge distribution and discuss the behavior at large values of \( z \). [20 points]

b) Verify that your solution fulfills Poisson equation for the given charge distribution. [20 points]
**Problem 3:** [40 points]

An infinite flat sheet of charge density per unit area $\sigma$, located in the xy plane, is forced to oscillate along the x-axis. The velocity of charges at time $t$ is given by $v = \hat{x}v_0 \cos(\omega t)$, resulting in electromagnetic radiation.

(a) Solve for all components of the electromagnetic fields everywhere. [20 points]
(b) How much energy per unit area is radiated away in a time T? (You may assume $T \gg 1/\alpha$.) [20 points]

**Problem 4:** [40 points]

A collimated beam of protons has the form of a very long cylinder of radius $R$. The speed of the protons is $v$ along the cylinder’s axis direction, and the number of protons per unit volume inside the cylinder is a constant, $n$. Suppose that the beam now leaves the region where it has been collimated by external forces, and enters a region of empty space.

(a) Find the electric and magnetic fields inside and outside of the beam. [16 points]
(b) What is the total force (magnitude and direction) on a proton within the beam at a distance $r$ from the beam axis? [12 points]
(c) Now suppose that a similarly collimated cylindrical antiproton beam of the same radius and number density is moving through the proton beam, with the same beam axis, but in the opposite direction with speed $v'$. What is the total force (magnitude and direction) on a proton at a distance $r$ from the beam axis? [12 points]

(Proton have charge $|e|$, antiprotons have charge $-|e|$.)
Problem 1: [40 points]

A metal bar of length $L$, mass $m$, and resistance $R$ is placed on frictionless rails that are inclined at an angle $\theta$ above the horizontal. The rails have negligible resistance. A uniform magnetic field is directed downward as shown above. The bar is released from rest and slides down the rail. The acceleration due to gravity is $g$.

(a) Is the direction of the induced current in the bar from $a$ to $b$ or from $b$ to $a$? [10 points]

(b) What is the terminal speed of the bar? [10 points]

(c) What is the velocity of the bar as a function of time? [20 points]

Problem 2: [40 points]

The plane $z = 0$ in a Cartesian coordinate system carries a surface charge density of the form $\sigma = \sigma_0 (1 - \cos(\alpha x)\cos(\beta y))$. Here, $\alpha$ and $\beta$ are real (as opposed to complex) constants.

a) Find throughout space the electrostatic potential due to this charge distribution. [20 points]

b) Check that your solution satisfies the Poisson equation for $z = 0$. [15 points]

c) Discuss the potential at large distance from the $xy$-plane. [5 points]
**Problem 3:** [40 points]

A liquid of permeability $\mu$ and mass density $\rho$ is contained in a thin U-tube of circular cross-section. Consider the liquid as a linear paramagnet. When the tube is inserted between the flat poles of a permanent magnet, the liquid between the poles rises by a height $h$ (see the figure). You may neglect all end effects. The initial field between the poles is $H_0$, the permeability of the tube material is $\mu = 1$, and the inner radius of the tube is $a$.

a) Find an expression for $h$ in terms of the magnetization $M$ in the liquid and $H_0$ and the acceleration due to gravity $g$. [13 points]

b) By solving the appropriate magneto-static boundary value problem express $M$ in terms of $H_0$ and derive an expression for $h$ in terms of $H_0$. [20 points]

c) A paramagnetic fluid has mass density $\rho$ of 2 g/cm$^3$ and a permeability of 99. How high (approximately) will the fluid rise if $\mu_0 H_0 = 0.1$ T? [7 points]

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**Problem 4:** An infinitely long solid insulating cylinder [40 points]

An infinitely long solid insulating cylinder has radius $a$ and a uniform constant charge density $\rho$. It is rotating around the $\hat{z}$ axis, which is its axis of symmetry, in the $+\phi$ direction with angular velocity $\omega$. The cylinder has the same permittivity and permeability as vacuum.

(a) Find the magnetic field $\mathbf{B}$ everywhere. [12 points]

Now assume that $\omega = \omega_0 + \alpha t$ is a slowly increasing function of time.

(b) Find the total electric field $\mathbf{E}$ everywhere. [16 points]

(c) Suppose that the cylinder is surrounded by a thin circular loop of wire of radius $b$ and resistance $R$. Find the current induced in the wire. [12 points]
Problem 1: Motion of a point charge under the uniform static magnetic field [40 points]
Consider a point positive charge with mass $m$ and charge $e (>0)$ moving in a uniform static magnetic field $\mathbf{B}$ parallel to the $z$-axis. The initial conditions for the velocity in Cartesian coordinates is at $t = 0$ are $v_x(0) = v_0, \quad v_y(0) = 0, \quad v_z(0) = 0$.

a) Express the equation of motion in vector form as well as its components in Cartesian coordinates. [10 points]

b) Using the initial conditions, find the components of the velocity of the point charge as a function of time. Evaluate the change of its kinetic energy and explain why. [15 points]

c) Given the initial conditions in the $z = 0$ plane are $x(0) = y(0) = z(0) = 0$. Express the trajectory of the point charge, its radius and periodicity. [15 points]

Problem 2: A very long, metallic, grounded waveguide [40 points]
Consider a very long, metallic and grounded waveguide with a square shaped inner cross section of side $D$. A point charge $Q$ is suspended on the axis of this guide at a location far from either end.

a) Find the electrostatic potential everywhere inside the guide. [25 points]

b) What is the asymptotic form of the potential at locations far from the point charge? [10 points]

c) Make a sketch of the electric field lines in a region far from the point charge. [5 points]
Problem 3: Magnetic field induced between circular parallel capacitor plates [40 points].

When a charged circular parallel plate capacitor is discharged by connecting a conducting wire as shown in Figure 3, a magnetic field $\mathbf{B}$ is induced. The radius of the parallel plates is $R$ and the distance between the parallel plates is $d$ ($d \ll R$). $I(t)$ is the electric current on the conducting wire. We neglect the effects of stray fields at the edges of the plates and of radiation.

Figure 2

a) Express the displacement current density in terms of the electric current $I(t)$ on the conducting wire [15 points]

b) Find $\mathbf{B}(\mathbf{r},t)$ between the plates, far from the edges. [25 points]

Problem 4: Long coaxial cable [40 points]

Consider a long coaxial cable consisting of an inner cylinder with radius $a$ and an outer cylinder of radius $b$. Suppose that the electric field is given in cylindrical coordinates $(r,\phi,z)$ in the region $a < r < b$ by

$$\mathbf{E}(r,\phi,z) = N r^n \cos(kz - \omega t) \hat{r}$$

where $N$, $k$, and $\omega$ are positive constants and $n$ is a rational number. The region between the cylinders has the same permittivity and permeability as vacuum.

(a) Solve for $n$ and determine the magnetic field for $a < r < b$. [16 points]

(b) Find the charge density and the current density on the inner cylinder. [8 points]

(c) Find an expression for the vector potential $\mathbf{A}$ in the region $a < r < b$. [8 points]

(d) Find the total time-averaged electromagnetic power transmitted by the cable. [8 points]
Problem 1: Electrostatic potential [40 points]

Consider an electrostatic potential \( \phi(r) = \frac{A e^{-kr}}{r} \):

a) Using the Poisson equation, find the charge density distribution in the space except at the origin. (15 points)

b) The fact that the potential becomes infinity at the origin means the existence of a point charge at the origin. First obtain the electric field from the potential \( \phi(r) \), then find the magnitude of the point charge and its sign. (15 points)

c) Find the total charge distributed other than the origin, also indicate its sign. (10 points)

Problem 2: A uniformly polarized ferroelectric slab is inserted into a parallel plate capacitor [40 points]

A uniformly polarized ferroelectric slab is inserted into a parallel plate capacitor as shown in the Figure. The capacitor area is \( A \), and the lateral sizes are much larger than \( a, b \). The two plates of the capacitor are connected through a thin wire.

a) Neglecting edge effects, determine the electric field everywhere in the capacitor. [16 points]

b) What is the electric field inside the ferroelectric plate for \( b \gg a \), and for vanishing \( b \)? [8 points]

c) Determine the force vector acting on the upper capacitor plate. [16 points]
Problem 3: A spherical shell with inner radius \( a \) and outer radius \( b \) with a uniform, fixed magnetization \( M \) [40 points].

A spherical shell with inner radius \( a \) and outer radius \( b \) carries a uniform, fixed magnetization \( M \) as shown in the figure. Using the method of separation of variables, find the vectors of the magnetic field and of the magnetic induction at:

a) \( r < a \), [8 points]

b) \( r > b \), [16 points]

c) \( a < r < b \). [16 points]

Hint: a solution for the Laplace’s equation in spherical coordinates with azimuthal symmetry and \( z \)-axis along the magnetization direction.

Then the general solution (magnetic scalar potential)
is \( \Phi_m(r,\theta) = \frac{U(r)}{r} P(\theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos(\theta)) \).

Figure 2

Problem 4: Long concentric conducting tubes [40 points]

Consider a set of long concentric conducting tubes (length \( l \)) sharing the same axis along the tubes shown in Figure 3. The radii of the tubes are \( a_1 \) and \( a_2 \), respectively. At one end these tubes are connected via a resistor while the other ends are connected to a power supply, \( \Delta \phi \).

(a) Find the magnetic field induced by the currents with the same magnitude but with the opposite directions.

(b) Find the self-inductance of this circuit. [10 points]
Problem 1: The potential inside and outside of a conducting sphere containing a line charge [40 points]

A line charge of uniform density $\lambda$ is placed radially inside a grounded conducting spherical shell of inner radius $a$, the two ends of the line charge being at distance $r_1$ and $r_2$ from the center of the sphere.

a. (20 pts) Determine the image charge.
b. (20 pts) What is the potential $\phi$ outside and inside the sphere?

Hint: $\int dx / \sqrt{1 + x^2 - 2cx} = \ln(-c + x + \sqrt{1 + x^2 - 2cx})$

Problem 2: A large conducting slab with a small hole [40 points]

A small cylindrical hole of radius $a$ is made near the center of a large conducting slab of uniform conductivity $\sigma$ carrying - prior to the introduction of the hole - a current of uniform density $J_0$.

The lateral size of the slab is considered large enough that effects due to its sides can be neglected. Furthermore, its thickness is large enough that any $z$-axis dependence at the top and bottom faces can be neglected as well.
a) (20 pts) Determine the electrostatic potential distributions in the slab and in the hole, respectively.
b) (10 pts) Find the electric field in the hole.
c) (10 pts) The space of the hole is filled with a material of conductivity \( \sigma' \). Determine the current density in this space.

**Problem 3: A cylinder filled with the uniform magnetization [40 points]**
Consider a cylinder of length \( L \) and radius \( R \) centered at the origin and aligned the \( z \)-axis, filled with the uniform magnetization \( \mathbf{M} = M\hat{z} \).

(a) (15 pts) Show that the magnetic scalar potential along the \( z \)-axis is

\[
\Phi_M = \frac{1}{2} M \left[ \sqrt{\left( z - \frac{1}{2} L \right)^2 + R^2} - \sqrt{\left( z + \frac{1}{2} L \right)^2 + R^2 + 2z} \right]
\]
inside the cylinder \((-\frac{1}{2} L < z < \frac{1}{2} L)\)

(b) (5 pts) What is the magnetic scalar potential along the \( z \)-axis outside the cylinder?
(c) (15 pts) What is the \( \mathbf{B} \) field along the \( z \)-axis inside the cylinder?
(d) (5 pts) What is the \( \mathbf{B} \) field along the \( z \)-axis outside the cylinder?

**Problem 4: The propagation of an electromagnetic wave [40 points]**
Consider the propagation of an electromagnetic wave with peak electric field \( E_0 \) and angular frequency \( \omega \) between two parallel perfectly conducting infinite plates spaced a distance \( d \) apart. One plate occupies the \( x = 0 \) plane and the other the \( x = d \) plane. Take the wave to propagate in the \( \hat{z} \) direction, and consider only the case that the electric field \( \mathbf{E} \) is everywhere parallel to the plates.

(a) Find expressions for the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) between the plates. [25 points]
(b) Find the minimum angular frequency \( \omega_{\min} \) below which waves of this type will not propagate without attenuation. [10 points]
(c) Find the exponential attenuation length for waves with angular frequency below \( \omega_{\min} \). [5 points]
**Problem 1: The electric field inside a dielectric sphere** [40 points]

A dielectric sphere, of radius $a$, has a uniform polarization $P$, find the electric field inside the sphere $\mathbf{E}_{\text{polar}}(\mathbf{r})$ with the following steps. Consider a model that the dielectric sphere for the positive and negative charge.

(a) Find the electric field $\mathbf{E}_{\text{general}}(\mathbf{r})$ inside a sphere, of radius $a$ with a uniform charge density $\rho$. Take the center of the sphere as the origin. [20 points]

(b) Consider two spheres of radius $a$, one with its center at $+u/2$ from the origin with charge density $+\rho$, the other with its center at $-u/2$ from the origin with charge density $-\rho$. Find the electric field $\mathbf{E}_+(\mathbf{r})$ and $\mathbf{E}_-\mathbf{r}(\mathbf{r})$ for each sphere. [10 points]

(c) The electric field $\mathbf{E}_{\text{polar}}(\mathbf{r})$ from the above $\mathbf{E}_+(\mathbf{r})$ and $\mathbf{E}_-(\mathbf{r})$, and then express $\mathbf{E}_{\text{polar}}(\mathbf{r})$ with the polarization vector $P$, assuming $|u| << a$. [10 points]

**Problem 2: A very long solenoid** [40 points]

A very long solenoid, of radius $a$, has its axis of symmetry along the $z$-axis. The solenoid has $N$ turns of wire and total length $l$. The wire carries a slowly time-varying current $I_s$, so that $dI_s/dt \neq 0$. (You may consider only regions in the solenoid but very far from the ends of the solenoid.

Give both magnitude and direction for vector quantities.)

(a) Find the electric field both inside and outside of the solenoid. [15 points.]

(b) Find the Poynting vector both inside and outside of the solenoid. [10 points]

(c) Consider an imaginary cylinder, coaxial with the solenoid, with length $d$ and radius $r$ with $r < a$. Find the rate at which energy is flowing into this cylinder from the outside. [10 points]

(d) Repeat part (c), but now with $r > a$. [5 points]
Problem 3: Radiation associated with a particle moving at low velocity ($\beta<<1$) [40 points]

Consider the Lienard-Wichert formula in SI units:

$$\hat{E} = \frac{q}{4\pi \varepsilon_0} \left\{ \frac{n - \beta}{\gamma^2 R^3 (1 - \beta \cdot n)^3} \right\}_{ret} + \frac{1}{c} \left[ \frac{n \times [n - \beta \times \beta]_r}{R (1 - \beta \cdot n)^3} \right]_{ret}.$$

(a) Briefly explain the use of this equation: what does it compute? What are the meanings of all the symbols it incorporates? [7 points]
(b) Discuss the physical meaning of the two terms in the curly brackets. [5 points]
(c) Write down the corresponding expression for the magnetic field as a function of $\hat{E}$. [5 points]
(d) We now consider the limit of very small velocities ($\beta<<1$):
   i. Give the expression of the radiated electric field. [6 points]
   ii. Give the corresponding expression for the radiated magnetic field. [4 points]
   iii. Compute the Poynting vector associated to the radiated fields. [5 points]
   iv. Deduce the total radiated power. [8 points]

Problem 4: Optics in a doubly-negative (left handed) meta-material [40 points]

In man-made dielectric and magnetic material unusual situations arise when the electric permittivity and magnetic permeability are both real and negative. We consider a plane wave with electric field $\mathbf{E} = \mathbf{E}_0 \exp[-i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ and magnetic field $\mathbf{H} = \mathbf{H}_0 \exp[-i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$.

1. Let first consider a “standard” material with permittivity $\varepsilon$ and magnetic permeability $\mu$ both real and positive. Find the Maxwell equations satisfied by $\mathbf{E}_0$ and $\mathbf{H}_0$ in such a medium, the velocity of light in such a medium, and what combination of $\mathbf{E}_0$, $\mathbf{k}$ and $\mathbf{H}_0$ form a right-handed set. [5 points]
2. From now on we consider a doubly-negative meta-material and write $\varepsilon = -|\varepsilon|$ and $\mu = -|\mu|$. Write the Maxwell equations satisfied by $\mathbf{E}_0$ and $\mathbf{H}_0$ in the meta-material and show that the role of $\mathbf{E}_0$ and $\mathbf{H}_0$ are interchanged compared to the “standard” case explored in (1). [10 points]
3. What can you state about the set $(\mathbf{E}_0, \mathbf{H}_0, \mathbf{k})$? [5 points]
4. We now specialize to a plane wave propagating along the $z$ axis and take $\mathbf{E}_0 = \mathbf{E}_0 \exp[-i(kz - \omega t)]\hat{z}$ and $\mathbf{H}_0 = \mathbf{H}_0 \exp[-i(kz - \omega t)]\hat{y}$ (where $\hat{u}$ stands for the unit vector along the $u$ direction). Find the Poynting vector $\mathbf{S}$ in the meta-material and compare its direction to $\mathbf{k}$. [10 points]
5. Discuss the implication of (4) to the sign of the index of refraction. Consider an incoming optical ray propagating at the interface of a “standard” material (with index of refraction $n_i>0$) and a meta-material (with index of refraction $n_r=-|n_z|<0$). Take the incident and refracted angles to be respectively $\theta_i$ and $\theta_r$. Write Snell’s refraction law at the interface and draw a schematic stressing the difference(s) with the usual situation of refraction at the interface between two standard materials. [10 points]
Problem 1: A very long hollow cylindrical shell [40 points]

A very long hollow cylindrical shell of radius $a$ is centered on the z-axis and carries surface charge density $\sigma = \sigma_0 \cos(\phi)$.

a) Find the electrostatic potential inside the shell. [20 points]

b) Find the electrostatic potential outside the shell. [20 points]

Problem 2: dipole located inside a grounded conducting spherical shell [40 points]

A point dipole $p$ is located at position $r$ on the $x$-axis inside a grounded conducting spherical shell of radius $R$ as shown in the figure. We choose the coordinate system such that the dipole lies in the $xz$-plane.

Determine the (vector) electric field at the center of the shell for:

a) $\alpha = 0$ deg. [20 points]

b) $\alpha = 90$ deg. [20 points]

![Fig. 1 A dipole located inside a grounded conducting shell](image)

Hint: Consider the dipole $p$ as composed of two charges $+q$ and $-q$ separated by a vector $a$ parallel to $p$ that is centered on $r$. Take the limit $a \to 0$ with the condition that $p = qa = constant$. 
**Problem 3: Vector potential of an infinitely wide flat conductive plate** [40 points]

The mathematical expression for the vector potential $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$ has a close resemblance to the expression for the electrostatic potential $\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$.

a) Find the electric field of the inside of a flat plate of a thickness $d$ with charges evenly distributed (the charge density $\rho$). [10 points].

b) Using the above resemblance between the expressions of the vector potential and the electrostatic potential, find the vector potential under the presence of a uniform current inside an infinitely wide flat conductive plate of a thickness $d$. [20 points]

c) Find the magnetic field $\mathbf{B}(\mathbf{r})$ for the above. [10 points]

**Problem 4: Diffraction & Faraday’s law** [40 points]

Consider a linearly polarized (along the $x$-axis) plane wave of frequency $\omega$ and amplitude $E_0$ incident on an opaque screen with a square aperture of edge size $a$.

a) Write down the electric and magnetic fields associated to this plane wave as a function of $\omega$ and $E_0$. [10 points]

b) By a simple application of Faraday’s law to a loop parallel to the screen on the side away from the source (Fig. 2), show that the wave has a longitudinal magnetic field component, $B_z$, after it has passed through the aperture. [15 points]

c) Show that the ratio $B_z/B_y$ is a measure of the diffraction angle. [15 points]

![Figure 1: geometry associated to problem 1. $k$ is the wavevector of the initial plane wave and $E_x$ and $B_y$ are the corresponding electric and magnetic fields. [Figure from K. T. McDonald and M. Zolotorev, Am. J. Phys. 68, 674 (2000)]]
Problem 1: An infinite flat sheet of uniform charge density [40 points]

An infinite flat sheet of uniform charge density $\sigma$, located in the x, y plane, oscillates along the x-axis. The velocity of the charges is given by $\vec{v} = v_0 \hat{x} \cos(\omega_0 t)$.

a. Find the electric and magnetic fields everywhere. [30 points]
b. What is the average power radiated per unit area of the sheet? [10 points]

Problem 2: Meissner effect [40 points]

The magnetic scalar potential outside a spinning shell (radius $R$, angular velocity $\omega$, total charge $q$) of charge is:

$$\Phi_{\text{out}} = -\frac{\mu_0 q R^2}{12 \pi r^2} P_1(\cos \theta)$$

a. Show that the magnetic field outside a spinning spherical shell of charge is

$$\vec{B}(\vec{r}) = -\nabla \Phi_M(\vec{r}) = \frac{\mu_0}{4 \pi} \left( \frac{3 \hat{m} \cdot \vec{r} - \hat{m}}{r^5} \right)$$

where the magnetic dipole moment of the spinning sphere is

$$\hat{m} = \frac{1}{3} q R^2 \vec{\omega}.$$ [20 points]

b. Below the critical magnetic field $H_c$, type I superconductors exhibit the Meissner effect, in which all magnetic fields are excluded from the volume of the superconductor. Consider a superconducting sphere of radius $a$ placed in an otherwise uniform magnetic induction $B_0$. Since the magnetic field is excluded from the sphere, the magnetic induction $B$ is tangent to the sphere at its surface as shown in Figure below.

i. Show that the total magnetic induction outside the sphere is

$$\vec{B}(\vec{r}) = B_0 + \frac{\mu_0}{4 \pi} \left( \frac{3 \hat{m} \cdot \vec{r} - \hat{m}}{r^5} \right)$$

where the induced dipole moment is

$$\hat{m} = -\frac{2 \pi a^3}{\mu_0} B_0.$$ [10 points]

ii. Calculate the energy required to place the sphere in the magnetic field.

[hint: the magnetic field induction exerts a pressure on the surface of the sphere. Compute $\delta W$ done on the field first when the radius of the sphere expands from $r$ to $r + \delta r$.] [10 points]
Problem 3: Space charge limited current in a diode [40 points]

Consider the space charge-limited flow of current in a planar vacuum diode. We assume that the cathode represents an unlimited source of electrons at zero velocity that are accelerated by the electric field toward the anode. The electrons in the anode-cathode gap form a space charge density
\[
\rho = \frac{J}{v}
\]
where \(J\) is the current density and \(v\) the electron velocity, that modifies the field and repels the electrons leaving the cathode. The current becomes limited when the electric field at the surface of the cathode vanishes.

a. Express the Poisson’s equation for this problem for non-relativistic motion of electrons. [20 points]

b. For nonrelativistic motion of the electrons, calculate the space charge-limited current in the diode. \(V\) is the potential of the anode relative to the cathode. The quantity
\[
K = \frac{4e_0 A}{9d^2} \frac{2e}{m}
\]
where \(A\) is the area of the cathode, \(d\) the anode-cathode separation, \(e = |q|\) the magnitude of the electron charge, and \(m\) the electron mass, is called the perveance of the diode. [20 points]

Problem 4: Paradox? [40 points]

Consider a point charge \(q\) moving at constant relativistic velocity \(v\) along the \(z\)-axis.

1. Write down the expression for the particle’s electric field in all direction perpendicular to its velocity. How does the field depend on \(\gamma\) the Lorentz factor? [10 points]

2. Compute the electric flux \(\Phi_E\) from the particle using Gauss’s law. [Hint: the integral
\[
\int_0^\pi \sin z \left[ \frac{1}{1 - \beta^2 \sin^2 z} \right]^{3/2} = 2\gamma^2 \text{ might turn out to be useful}…\].

3. Compare your answer from (1) and (2) and try to explain the apparent “discrepancy” in the \(\gamma\)-dependence. [10 points]
**Problem 1: Dielectrics [40 points]**

The space between two coaxial conducting cylinders of length $L$ is half-filled with a dielectric having relative dielectric constant $\varepsilon_r$. The cylinders have radius $r_1$ and $r_2$, as shown in fig, and are connected to a $V_1$ battery.

A. Find the fields $\vec{E}$ and $\vec{D}$ in the air and in the dielectric in the space $r_1 < r < r_2$. [20 points]

B. Find the surface charge induced on the inner conductor at points adjacent to the air, and at points adjacent to the dielectric. [10 points]

C. Find the total charge on the inner conductor, and the capacitance. [10 points]

**Problem 2: A cylindrical permanent magnet [40 points]**

A cylindrical permanent magnet of length $L$ and radius $a$ has uniform magnetization $M$ directed along the axis of the magnet (see Figure).

A. Find the magnetic field outside the magnet on the $z$ axis. [20 points]

B. Analyze the limit $z \gg a, L$ and compare the result to the field of a magnetic dipole. [20 points]
**Problem 3: Electromagnetic waves in a conductor? [40 points]**

We consider a homogenous and isotropous medium with \(\varepsilon_r \sim 1\) and \(\mu_r \sim 1\). We assume the medium to be neutral conductor which is well described by Ohm’s law with conductivity \(\sigma\).

A. Write the Maxwell’s equations satisfied by the electric \(\tilde{E}\) and magnetic \(\tilde{B}\) fields in the medium and deduce a second order partial derivative equation for \(\tilde{E}\). Show that when \(\sigma \to 0\) the usual vacuum wave equation is recovered. [10 points]

B. We now seek the solution of the latter equation by considering the electric field to be of the form \(\tilde{E} = E_0 \exp[i(\omega t - kx)] \hat{z}\), where \(E_0\) is a constant, \(\omega\) is a positive real number, and \(\hat{z}\) is the unit vector along the \(z\)-axis.
   a. Find the expression of \(k^2\) as a function of \(\omega, \sigma, \varepsilon_0\) (the vacuum electric permittivity), and \(c\) (the velocity of light). [10 points]
   b. Consider a Copper conductor, with a conductivity \(\sigma = 5.7 \times 10^8 \Omega^{-1}\).m\(^{-1}\). Find the frequency domain where \(\frac{\sigma}{\varepsilon_0 \omega} \gg 1\). Is this approximation applicable for typical AC electrical frequencies and typical frequencies associated to the visible spectrum? [10 points]
   c. Assuming \(\frac{\sigma}{\varepsilon_0 \omega} \gg 1\), compute \(k\) as a function of \(\omega, \sigma, \varepsilon_0\), and \(c\), and deduce the general form of the electromagnetic wave in the conductor and give the solutions for the electric and magnetic fields. [10 points]

**Problem 4: Electromagnetic field created by a moving charge [40 points]**

The figure below describes the trajectory of a particle with charge \(q > 0\) and constant velocity \((3/5)c\) (where \(c\) it the velocity of light). The change of velocity direction at \((x=0, y=0)\) is instantaneous. At \(t=0\), the charge is located at \((x=0, y=a)\).

A. At \(t=0\), what is the electric field at points \(P\) \((x=0, y=a)\) and \(Q\) \((x=2a, y=a)\) [a simple sketch showing the E-field directions at both points along with a qualitative (and clear) discussion of your sketch are required]. [20 points]

B. Plot the electric field lines at \(t=0\) in the \((x,y)\) plane. [10 points]

C. Plot the magnetic field lines at \(t=0\) in the \((x,y)\) plane. [10 points]
1. a. Two point charges, \(+q\) and \(-q\), are separated by a distance 2d. An isolated conducting spherical shell of outer radius \(a\) is inserted at the center point of the line joining the two charges. Determine the force acting on the charges for the case \(d >> a\).

![Diagram of two point charges and a spherical shell](image1.png)

b. An isolated conducting spherical shell of inner radius \(a\) is centered on the origin of a Cartesian coordinate system. The shell contains a thin ring of radius \(b\) located in the \(xy\)-plane and centered on the origin. The ring carries a line charge density \(\lambda\). Find the electric potential outside the shell at points \(P\) on the \(z\)-axis.

![Diagram of a spherical shell with a thin ring](image2.png)
2. A non-conducting liquid of dielectric constant $\varepsilon$ and mass density $\rho$ is contained in a U-tube of circular cross section with radius $a$. The dielectric constant of the tube material is $\varepsilon' = 1$. The tube is inserted in the middle between the plates of a parallel-plate capacitor (see Figure) which produces a fixed uniform electric field $E_0$ in the region far from the dielectric liquid. Separation $d$ and width (along the y-axis) of the plates are both much larger than $a$.

a) Neglecting all edge effects, determine the electric field inside and outside the liquid.

b) Find the height $h$ by which the liquid rises in terms of $E_0$. 

![Diagram of a U-tube with electric field and liquid level]
3. By shaping the pole faces of a cylindrical magnet, as shown in the figure in cross section, and applying a time varying current to the coils, a magnetic field

\[ \vec{B} = -\hat{z}B_0 \frac{a}{\sqrt{a^2 + r^2}} t, \quad (r < a) \]

\[ \vec{B} = 0, \quad (r > a) \]

can be produced for times \( t > 0 \). For \( t < 0 \), the field is \( B = 0 \). Find the electric field induced for 1) \( r < a \) and 2) \( r > a \) and all \( t \).

4. Resonant cavities operated on the \( TM_{010} \) mode are commonly used to accelerate charged-particles. The idealized right-circular cylinder cavity is not useful for accelerating charged particle since it does not include entrance/exit apertures to inject/extract the charged particles. In the following, we seek the form of the fields associated to the \( TM_{010} \) mode of a realistic resonant cavity that incorporates entrance and exit apertures (see Fig. 1a). We will work in cylindrical coordinate \( (r, \phi, z) \). The TM\(_{mnp}\) mode corresponds to an electromagnetic fields configuration where the longitudinal magnetic field, \( B_z \), vanishes. The subscripts \( mnp \) represent the number of "field antinodes" along respectively the \( \phi, r, \) and \( z \) directions.

a) From the cylindrical symmetry and the definition of the \( TM_{010} \) mode, what is the expected \( \phi \)-dependence of the fields? What are the non-zero \( E \) and \( B \)-field components?
b) Assume the axial E-field \[ E_z(r = 0, z) \] is zero at \( z = d/2 \) and \( z = -d/2 \) (see Fig. 1a). What is the general z-dependence for the axial electric field \( E_z(r = 0, z) \)?

c) From now we take \( d = \lambda / 2 \) where \( \lambda \) is the resonant wavelength of the cavity and let \( L(z) = E_z(r = 0, z) \). The z-component of the electric field at any location \((r, \phi, z)\) can be expressed in the form \( E_z(r, \phi, z) = T(r, \phi)L(z) \), where \( T(r, \phi) \) represent the transverse coordinate dependence of \( E_z \). Find the differential equation satisfied by \( T \) and the corresponding solution for \( T \).

d) Write down the expressions for \( E_z(r, \phi, z) \) and the other non-zero components of the \( E \) and \( B \)-fields. Expand these solutions to 1\(^{st}\) order in \( r \).

e) Although Fig. 1b seems to suggest the cavity with aperture is still a right circular cylinder this is not the case in practice: the ends of the cavity are not flat. Find the equation of the end plate \( r(z) \) in the vicinity of the aperture from the boundary equation satisfied by the E-field. Use the 1\(^{st}\) order expansion for the fields.

**Hints:** You may need some of the following.

The equation \( r^2 \frac{d^2 Z}{dr^2} + r \frac{dZ}{dr} + (k^2 r^2 - n^2)Z = 0 \), has the solution \( Z = J_n(kr) \) where \( J \) stands for the 1\(^{st}\) kind Bessel function. The equation \( r^2 \frac{d^2 Z}{dr^2} + r \frac{dZ}{dr} - (k^2 r^2 - n^2)Z = 0 \), has the solution \( Z = I_n(kr) \) where \( I \) stands for the modified Bessel function.

The Taylor expansions of \( J \) and \( I \) functions are:

\[
I_n(x) = \left( \frac{1}{2} x \right)^n \left[ 1 + \left( \frac{x^2}{4} \right)^n + \ldots \right], \quad J_n(x) = \left( \frac{1}{2} x \right)^n \left[ 1 + \left( -\frac{x^2}{4} \right)^n + \ldots \right].
\]

We also remind the identities \( \frac{d}{dx} (x I_1) = x I_0, \quad \frac{d}{dx} (x J_1) = x J_0, \quad \frac{d}{dx} J_0(x) = J_1(x) \) and \( \quad \frac{d}{dx} I_0(x) = I_1(x) \). The Laplacian in \((r, \phi, z)\) is \( \nabla^2 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) + \frac{1}{r^2} \frac{d^2}{d\phi^2} + \frac{d^2}{dz^2} \).
1. An isolated conducting sphere of radius $a$ is contained in an isolated spherical metallic shell of inner and outer radius $b$ and $b'$, respectively. The centers coincide.

With the boundary condition that at infinity the potential is zero find the electrostatic potential everywhere for

a) A charge $Q$ placed on the sphere
b) A charge $Q$ placed on the shell

2. An infinitely long cylinder with radius $a$ and permeability $\mu$ is placed with its axis into an initially uniform magnetic field $H_0$ with its axis perpendicular to $H_0$. Find the resultant field inside and outside of the cylinder.
3. Consider a region of empty space with a static magnetic field whose z component has the form $B_z = B_0 \left( 1 + \left( \frac{z}{\lambda} \right)^2 \right)$ where $B_0$ and $\lambda$ are constants. The field is cylindrical-symmetric about the z-axis. The transverse components, $B_x$ and $B_y$, are zero at the origin (z=0). A particle of charge $q$ and mass $m$ is initially started at the origin with speed $v \ll c$. For the value $\lambda = \infty$, the magnetic field is uniform and the trajectory of the particle assumes a periodic motion in x-y. For large values of $\lambda$, $B$ changes by a small fraction over a cycle of motion. Under such conditions, the quantity $S = \int (dxP_x + dyP_y)$ is constant (so-called adiabatic invariant). Here $P_x$ and $P_y$ are the transverse components of the canonical momentum and the integral is performed over one cycle of the x-y motion.

a) Find the flux of $B$ through the particle’s orbit projected onto the x-y plane in term of $S$.

b) Under the conditions described, the particle is confined within an interval $|z| < z_{MAX}$. Find the value of $z_{MAX}$ (as a function of initial angle $\theta$ and $\lambda$) for a particle whose initial velocity is directed at an angle $\theta$ from the z-axis; see Figure below.
4. A solid cylinder rod of radius $R$ and length $L$ is divided in half. One half of the rod is filled with a positive charge density ($+\rho$) and the other half with a negative charge density ($-\rho$); see Figure A (the charge density $\rho$ is a constant).

a) What is the dipole moment $\vec{p}$?

The cylinder is spun about an axis perpendicular to its length and passing through its center, with angular velocity $\omega$ (such that relativistic effects are ignorable); see Figure B.

b) How does the radiated power per unit solid angle far from the dipole, $\left\langle \frac{dP}{d\Omega} \right\rangle$ as a function of the observation angles. [Hint: recall that $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{4\pi c^3} (\vec{p} \times \vec{n})^2$; where $\vec{n}$ is the direction of observation, and $\vec{p} \equiv \frac{d^2\vec{p}}{dt^2}$. The observation angle is defined as $\theta \equiv \angle(\vec{n}, \hat{z})$].

c) What is the total radiated power?
Problem 1: An infinitely long line carries charge per unit length $\lambda$. It is placed parallel to and at a distance $d$ from the center of an infinitely long grounded conducting cylinder of radius $a$, where $a < d$.

(a) Calculate the force per unit length on the line charge.

(b) Calculate the surface charge density on the cylinder as a function of position on the cylinder.

Problem 2: Consider the propagation of an electromagnetic wave in vacuum between two infinite parallel conducting plates spaced a distance $d$ apart, occupying the planes $z = 0$ and $z = d$. For the case in which the electric field is everywhere parallel to the plates, and the wave propagates in the $x$ direction:

(a) Give an expression for the electric and magnetic fields between the plates.

(b) Show that there is a minimum angular frequency $\omega_c$ below which waves will not propagate, and derive an expression for $\omega_c$.

Problem 3: A very long solenoid has $n$ turns of wire per unit length, and carries a slowly time-varying current $I(t)$. The radius of the solenoid is $a$, and the axis of symmetry is the $z$ axis.

(a) Find the magnetic field $\vec{B}$ inside the solenoid.

(b) Use Faraday’s Law in integral form to find the electric field $\vec{E}$ both inside and outside of the solenoid.

(c) Consider an imaginary cylinder, coaxial with the solenoid, and with length $d$ and radius $r$. Find the rate at which electromagnetic energy is flowing into this cylinder from the outside, for the two cases $r < a$ and $r > a$. 
Problem 4: A coaxial cable consists of two infinitely long coaxial perfectly conducting cylinders with radii $a$ and $b$, with $a < b$, and with the common axis being the $z$ axis. The inner conductor carries a current $I$ in the $+\hat{z}$ direction, and the outer conductor carries the current back in the opposite direction. The region between the conductors has magnetic permeability $\mu$ (different from the permeability $\mu_0$ of empty space).

(a) Find the magnetic fields $\vec{B}$ and $\vec{H}$ everywhere.

(b) Find the self-inductance per unit length of the cable.
Problem 1. The half space $x > 0$ is filled with a constant magnetic field $\vec{B} = (0, 0, B_0)$, and the half space $x < 0$ is filled with a field in the opposite direction $\vec{B} = (0, 0, -B_0)$. An electron is shot out of the origin with initial velocity $\vec{v} = (-v_0/\sqrt{2}, -v_0/\sqrt{2}, 0)$. Describe its subsequent motion as quantitatively as possible.

Problem 2. A plane electromagnetic wave of angular frequency $\omega$ and peak electric field $E_0$ is moving in the $\hat{z}$ direction, and is polarized in the $\hat{x}$ direction. The wave is incident on a charge $Q$ of mass $m$. The charge is bound to the origin by a restoring potential

$$V(r) = m\omega_0^2 r^2/2,$$

where $r$ is the distance to the origin and $\omega_0$ is a constant. It is also acted on by a viscous damping force given by

$$\vec{F}_d = -m\omega_d \vec{v},$$

where $\vec{v}$ is the velocity of the charge, and $\omega_d$ is another constant.

(a) Find the position $\vec{r}$ of the charge as a function of time, ignoring magnetic effects. What is the phase angle between the electric field of the wave $\vec{E}(t)$ and $\vec{r}(t)$?

(b) Find the intensity of the scattered radiation as a function of direction and of $\omega$. (You do not need to evaluate the overall multiplicative factor.)
Problem 3. An azimuthally symmetric perfect conductor is approximately spherical, with mean radius $R_0$, in the sense that the radius $R$ as a function of the polar angle $\theta$ is

$$R(\theta) = [1 + \Delta(\theta)]R_0$$

with $\Delta(\theta) \ll 1$ and

$$\int_{-1}^{1} \Delta(\theta) d(\cos \theta) = 0.$$

This conductor is given a total charge $Q$. To first order in the function $\Delta$, find the potential outside the conductor.

(You may leave your result in terms of the Legendre polynomials $P_n(x)$, which satisfy:

$$\int_{-1}^{1} P_n(x)P_{n'}(x)dx = \delta_{nn'} \frac{2}{2n + 1}$$

for non-negative integers $n, n'$.)

Problem 4. The angular distribution of radiation emitted by an accelerated charged particle with charge $q$ and velocity $\vec{v} = c\vec{\beta}$ may be determined by

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \vec{\beta}) \times (d\vec{\beta}/dt')] \right|^2}{(1 - \vec{\beta} \cdot \hat{\mathbf{n}})^5}$$

where $\hat{\mathbf{n}}$ is the unit vector in the direction to the observer.

(a) What interpretation does $t'$ have?

(b) What approximations, if any, are inherent to this expression?

(c) Suppose the charge undergoes linear acceleration, so that $d\vec{\beta}/dt$ is parallel to $\vec{\beta}$. Let $\theta$ be the angle between $\vec{\beta}$ and $\hat{\mathbf{n}}$. Find the locations $\theta_{\text{max}}$ of the peak radiated intensity.

(d) How do these $\theta_{\text{max}}$ values depend on the particle’s total energy $E$ if the particle is highly relativistic?
Problem EM1. In a vacuum diode, electrons with mass \( m \) and charge \( q \) are "boiled" off a hot cathode, at potential \( V = 0 \), and accelerated across a gap to the anode, which is held at positive potential \( V = V_0 \). The cloud of moving electrons within the gap (called space charge) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current \( I \) flows between the plates.

Suppose the plates are large relative to the separation (area \( A \gg d^2 \) in the figure below), so that edge effects can be neglected. Then the potential \( V \), the charge density \( \rho \), and the speed of the electrons \( v \) are all functions of \( x \) only.

(a) Assuming the electrons start from rest at the cathode, what is their speed at point \( x \), where the potential is \( V(x) \)?

(b) In the steady state, the current \( I \) is independent of \( x \). What, then, is the relation between \( \rho \) and \( v \)?

(c) Use the results above and Poisson's equation to obtain a differential equation for the potential \( V(x) \), not involving \( \rho \) or \( v \), and solve it. (Hint: try a power law form.)

(d) Show that a non-linear relation holds between \( I \) and \( V_0 \), of the form

\[
I = K V_0^n
\]

where \( K \) is a constant that you will find in terms of the given quantities, and \( n \) is a rational number.
Problem EM2. Consider the right circular cylindrical cavity pictured below. The inner and outer walls of the cavity, and the ends, are perfect conductors. The inner conductor has radius $a$, the outer conductor has radius $b$, and they are coaxial with the $z$-axis. The height of the cavity is $h$.

Consider the TEM modes of the cavity (the modes that have an electric field with no component in the $z$ direction), in the vacuum region $a \leq r \leq b$ and $0 \leq z \leq h$, with peak electric field $E_0$. Find expressions for the electromagnetic fields and the resonant frequencies and wavelengths of all such modes in terms of the given quantities.

Problem EM3. A long solenoid consists of $N$ turns of wire with resistance $R$ wound on a right circular cylinder of length $d$ and radius $a$, with $d \gg a$.

(a) Suppose that a slowly time-varying current $I(t) = I_0 \cos(\omega t)$ flows in the solenoid. Find the magnetic and electric fields everywhere inside and outside of the solenoid.

(b) A circular ring of wire with radius $b$ is coaxial with the solenoid and outside of it ($b > a$). Suppose now that a slowly varying current $I(t) = I_0 \cos(\omega t)$ flows in the ring. What current is induced in the solenoid?
Problem EM4. An electromagnetic wave is propagating in a non-conducting linear medium that has the same permeability as vacuum, \( \mu = \mu_0 \). The magnetic field of the wave is given by

\[
\vec{B} = B_0 \left( \frac{\hat{y} + \hat{x}}{\sqrt{2}} \right) \cos \left( \frac{\sqrt{\omega} c}{2c} x + ay - \frac{\omega}{2c} z + \omega t \right)
\]

where \( a \) is a positive constant, and \( c \) is the speed of light.

(a) Find \( a \). What is the direction of propagation of the wave?

(b) What are the index of refraction and the permittivity of the material?

(c) Find the electric field \( \vec{E} \) of the wave.

(d) The wave is incident on a square with side \( d \) of some perfectly absorbing material that lies in the \( y = 0 \) plane. How much energy is absorbed by the material in time \( T \)? (Assume \( T \gg 1/\omega \).)
Problem EM1. Two conducting coaxial cylinders are placed in a uniform magnetic field $B$, which is parallel to the axes of the cylinders. The inner cylinder has radius $R_1$ and is grounded. The outer cylinder has radius $R_2$ and is held at a positive potential $V$. Electrons are released from the inner cylinder with negligible velocity, and follow curved paths toward the outer cylinder.

(a) Find the system of differential equations governing the coordinates $(r, \phi, z)$ of an electron as a function of time.

(b) For a given value of $B$, what is the minimum potential $V$ that will cause an electron to reach the outer cylinder?

Problem EM2. A long straight coaxial cable consists of an inner cylinder of radius $a$ and an outer cylinder of radius $b$. Both cylinders are perfect conductors, and the volume between them is a vacuum. Consider TEM propagation, i.e. a pure transverse wave traveling down the cable. Take the angular frequency to be $\omega$, and the peak electric field strength to be $E_0$.

(a) What are the magnetic and electric fields inside the cable, as a function of $r$, the distance from the center?

(b) What is the time-averaged power transmitted by the cable?
Problem EM3. A microwave antenna radiating at 10 GHz is to be protected from the weather by a flat plastic shield with dielectric constant 2.5 times that of vacuum, no magnetic properties, and a thickness $d$ (to be determined).

\[ \begin{array}{c}
\text{air} \\
\text{plastic} \\
\text{air}
\end{array} \]

(a) Assuming normal incidence, what are the boundary conditions to be satisfied at the air-shield boundaries?
(b) From these results, determine the minimum thickness of the shielding that will allow perfect transmission.

Problem EM4. A point dipole $\vec{p}$ is located at the center of a spherical cavity of radius $a$. The surface of the sphere is a conductor at zero potential.
(a) Find the function (a solution to Laplace's equation) that must be added to the dipole potential to satisfy the boundary conditions.
(b) Evaluate the induced charge density on the inside surface of the cavity.
(c) Find the dipole moment of the induced charge density.
NIU Ph.D. qualifier examination 2003 Fall (September 6, 2003)
Electricity and Magnetism
Solve 3 out of 5 problems.

I. Two infinite plane grounded conducting sheets intersect at an angle of \( \pi/3 \). A charge \( q \) is placed equidistant from the conductors and at a distance \( a \) from their line of intersection. Sketch the E field in the region around charge \( q \), between its two adjacent conducting sheets. Find the force exerted on the charge by the conducting sheets.

![Diagram of two infinite plane grounded conducting sheets intersecting at an angle of \( \pi/3 \). A charge \( q \) is placed equidistant from the conductors and at a distance \( a \) from their line of intersection.]

II. A linear dielectric sphere of radius \( a \) and dielectric constant \( \kappa \) carries a uniform charge density \( \rho \), surrounded by vacuum.
   a) Find \( \mathbf{E} \) and \( \mathbf{D} \) inside and outside the sphere.
   b) Find the energy \( W \) of the system.

III. An infinite flat sheet of charge density per unit area \( \sigma \), located in the xy-plane, is forced to oscillate along the x-axis. The velocity of charges at time \( t \) is given by \( \mathbf{v} = \hat{x}v_0 \cos(\omega t) \), resulting in electromagnetic radiation.
   (a) Solve for all components of the electromagnetic radiation.
   (b) How much energy per unit area is radiated away in a time \( T \)? (You may assume \( T \gg 1/\omega \).)

IV. The electric field \( \mathbf{E} \) and the electric displacement \( \mathbf{D} \) in a certain linear anisotropic medium are related by an effective dielectric tensor \( \epsilon_y \):

\[
\epsilon_y = \epsilon_0 \begin{pmatrix} 1 & ia & 0 \\ -ia & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
\mathbf{D}_i = \sum_{j=1}^{3} \epsilon_y E_j,
\]

\[
\epsilon_0 = \begin{pmatrix} 1 & ia & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
where \( a \) is a positive constant less than 1. The indices 1, 2, 3 correspond to \( x, y, z \) respectively. The material is non-conducting and has the same magnetic permeability as vacuum (\( \mu = \mu_0 \)).

(a) Given a plane wave propagating in the \( z \)-direction, find the polarization for which the medium has a definite index of refraction \( N (N = ck/\omega) \), and the corresponding values of \( N \).

(b) A semi-infinite slab of the above material fills the region \( z > 0 \). An electromagnetic plane wave of frequency \( \omega \) is incident normally on the flat direction of the material, and is therefore propagating in the \( z \)-direction. The wave enters the material from vacuum. The incident wave is linearly polarized with the electric field along the \( x \)-axis. Find the polarization of the reflected wave.

V. Suppose a photon of wavelength \( \lambda \) strikes a stationary electron and "bounce off" with a wavelength \( \lambda' \) at angle \( \theta \). This process is called Compton scattering; photons are neither created nor destroyed.

(a) Derive the Compton scattering formula relating \( \lambda', \lambda \) and \( \theta \).

(b) What part of the electromagnetic spectrum would best be used for experimental verification of Compton scattering, and why?

Electricity and Magnetism

Solve 3 out of 5 problems.

I. Find the potential \( \Phi \) and field \( \vec{E} \) for an unchanged conducting sphere placed in an initially uniform electric field, using an expansion in Legendre polynomials. Choose the \( z \) axis to be the initially uniform field direction.

a. Write the most general solution to Laplace’s equation in terms of the radial functions

\[
\frac{U(r)}{r} = A_i r^i + B_i / r^{i+1}
\]

and the Legendre polynomials \( P_i(\cos \theta) \). Since the problem has azimuthal symmetry, no spherical harmonics are required.

b. Use boundary condition at \( r = \infty \) to determine all the \( A_i \).

c. Use boundary condition at \( r = a \) to determine all the \( B_i \) except \( B_0 \). What determines \( B_0 \)?

d. With all coefficients determined, write explicit forms for \( \Phi \) and \( \vec{E} \) for the space outside the sphere.

e. Determine the induced charge density on an initially uncharged sphere.

f. Determine the total induced charge on an initially uncharged sphere.

II. A spherical shell of radius \( R \) carries a uniform surface charge density \( \sigma \). Calculate the vector potential \( \vec{A} \) and the magnetic field \( \vec{B} \), which are created when the sphere is rotating with an angular speed \( \omega \).

III. Classical model of Zeeman effect.

a. Consider an electron that executes 3-dimensional simple harmonic motion (SHM), i.e., it is subject to the potential per unit mass,

\[
V(x,y,z) = \frac{1}{2} \omega_0^2 \left( x^2 + y^2 + z^2 \right).
\]

Now turn on an external magnetic field \( \vec{B} = B \hat{z} \), where \( B \) is a constant. Show that the frequency of vibration is modified such that (in mks units)

\[
\omega' = \omega \sqrt{\frac{\omega_0^2}{\omega_0^2 + \left( \frac{eB}{2m} \right)^2}} = \frac{eB}{2m}; \quad \omega' = \omega_0;
\]

wherein \( e = 1.6 \times 10^{-19} \), \( e \) is the electron charge, and \( m = 9.1 \times 10^{-31} \) kg is the electron rest mass.
b. Suppose SHM is used as a (crude!) classical model of an electron’s orbit in an atom, and supposed the frequency shift is small compared to $\omega_0$. Then what is the corresponding splitting $\Delta E_\pm$ of the atomic energy level?

c. What percent change does a $B = 1$T field impart to the ground-state energy $E_0 = 13.6$ eV of the hydrogen atom? [Note: 1 eV/$h = 2.72 \times 10^{14}$ Hz, where $h$ is Planck’s constant.]

d. Is SHM an adequate model of the electron’s orbit? If so, why? If not, why not?

IV. Electrodynamical Lagrangian and Hamiltonian

a. Ascertain whether the Lagrangian

$$L(x,v,t) = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + e\mathbf{A} - e\Phi(x),$$

where $\mathbf{A}$, $\Phi$ are the vector and scalar potentials, respectively, yields the Lorentz force equation in mks units.

b. Construct the Hamiltonian associated with the Lagrangian of part (a).

V. Electron Synchrotron

The rate at which a relativistic, accelerating electron (charge $e = 1.6 \times 10^{-19}$ C, rest mass $m = 9.1 \times 10^{-31}$ kg) radiates energy is, in mks units,

$$P = \frac{2}{3} \frac{e^2}{4\pi e_0 c} \gamma^4 [\beta^2 - (\hat{\beta} \times \hat{\beta})^2],$$

in which $e_0 = 8.854 \times 10^{-12}$ F/m is the permittivity of free space, and $c = 3 \times 10^8$ m/s is the speed of light. Estimate the maximum beam energy that can viably be achieved using an electron synchrotron (a circular accelerator).