You may solve ALL FOUR problems, if you choose. Only the THREE BEST PROBLEM GRADES count towards your score. 
Total points on each problem = 40. Total possible score = 120.

**Problem 1.** Three masses $m_1$, $m_2$, and $m_3$ are suspended as shown in a uniform gravitational field with acceleration $g$. Masses $m_1$ and $m_3$ are connected by an unstretchable string which moves over a massless pulley whose center does not move. Mass $m_2$ is suspended from mass $m_1$ by a spring with constant $k$ and unstretched length $L$. Call $x$ and $z$ the heights of masses $m_1$ and $m_2$, respectively, as shown.

(a) Find the Lagrangian of the system, using $x$ and $y = z - x - L$ as your configuration variables. [10 points]

(b) Find the equations of motion of the system. [10 points]

(c) Solve for the relative motion of the masses $m_1$ and $m_2$, and in particular find the angular frequency of oscillation [10 points]

(d) What is the oscillation frequency if $m_1 \gg m_2, m_3$? Show using your answer above and/or physical arguments. [5 points]

(e) What is the oscillation frequency if $m_2 \gg m_1, m_3$? Show using your answer above and/or physical arguments. [5 points]

**Problem 2.** A point particle of mass $M_1$ approaches a massive sphere of radius $R$ and mass $M_2$. The two masses attract each other according to Newton’s law of gravitation. When very far away from each other, $M_1$ has initial speed $v_0$ while the sphere $M_2$ is initially at rest.

(a) If the initial impact parameter distance is $b$, find the distance of closest approach of the centers of the two objects. [20 points]

(b) Find the cross-section for the two objects to collide. [12 points]

(c) Give your answers and physical explanations for the limits $v_0 \to 0$ and $v_0 \to \infty$. [8 points]
Problem 3. A thin rod of length $\ell$ and mass $M$ is supported at one end by a smooth floor on which it slides without friction. The rod falls in a uniform gravitational field with acceleration $g$, starting from rest with an initial angle $\theta_0$ relative to the horizontal, as shown.

(a) Find the Lagrangian for the system. [10 points]
(b) Find the Hamiltonian for the system. [6 points]
(c) Find the time needed for the rod to fall to the floor. You may leave your answer in terms of a definite integral. [18 points]
(d) How far does the lower end of the rod move during this time? [6 points]

Problem 4. A water drop falls vertically in a uniform gravitational field $g$, growing by accretion of much smaller micro-droplets of water in the atmosphere. The micro-droplets have negligible velocity and air resistance is neglected. As the falling drop grows, it maintains a perfectly spherical shape, with radius we will call $R$. The mass density of water is a constant $\rho$, and the density of the micro-droplets (per volume of atmosphere) is $\epsilon \rho$, so that the mass of the falling drop increases at a rate given by its speed multiplied by $(\epsilon \rho)(\pi R^2)$.

(a) Find a relation between the speed of the falling water drop and the time derivative of its size, and use it to show that $R(t)$ obeys the differential equation

$$\frac{d^2 R}{dt^2} + \frac{n}{R} \left( \frac{dR}{dt} \right)^2 = C,$$

where $n$ is a certain integer that you will determine, and $C$ is a constant that you will find in terms of the given quantities. [20 points]
(b) Find a general relation for the downward acceleration of the drop in terms of $g$, $\epsilon$, and the instantaneous values of its speed $v$ and radius $R$. [8 points]
(c) Starting from the results found in part (a), try a power law solution of the form $R = At^\alpha$, and solve for the constants $A$ and $\alpha$, to show that for large $t$ the downward acceleration of the drop approaches the constant $g/N$, where $N$ is another integer that you will determine. [12 points]