Do ONLY THREE out of the four problems. Total points on each problem = 40.

**Problem 1.** A coupled oscillator consists of two springs with equal spring constants $k$ and two equal masses $m$, which hang from a fixed ceiling in a uniform gravitational field with acceleration $g$. The system oscillates in the vertical direction only.

(a) Find the Lagrangian for the system. [10 points]
(b) Find the angular frequencies of the normal modes of oscillation. [20 points]
(c) In the slower mode, find the ratio of the amplitude of oscillation of the upper mass to that of the lower mass. [10 points]

![Diagram of a coupled oscillator with two springs and two masses](attachment:image.png)

**Problem 2.** A particle of mass $m$ moves along the $x$-axis under the influence of a time-dependent force given by $F(x, t) = -kxe^{-t/\tau}$, where $k$ and $\tau$ are positive constants.

(a) Compute the Lagrangian function. [7 points]
(b) Find the Lagrangian equation of motion explicitly. [7 points]
(c) Compute the Hamiltonian function in terms of the generalized coordinate and generalized momentum. (Show clearly how you got this.) [7 points]
(d) Determine Hamilton’s equations of motion explicitly for this particular problem (not just general formulae). [7 points]
(e) Does the Hamiltonian equal the total energy of the mass? Explain. [6 points]
(f) Is the total energy of the mass conserved? Explain. [6 points]
Problem 3. Consider a particle of mass $m$ moving in a three-dimensional central potential

$$V(r) = -K/(r - R),$$

where $K$ and $R$ are fixed positive constants. Let the angular momentum of the particle be $\ell$.

(a) Find the differential equation of motion for $r$, the radial coordinate of the particle. [10 points]

(b) Derive an algebraic equation for the possible values of the radius $r_c$ of a circular orbit. (You do not need to solve the equation.) [10 points]

(c) Consider a small perturbation of a circular orbit, $r = r_c + \epsilon$. Find a linearized differential equation for $\epsilon$, written in terms of $r_c$, $R$, and $K$ (with $\ell$ eliminated). Use it to show that circular orbits are stable if either $r_c < R$ or $r_c > nR$, where $n$ is a certain integer that you will find. [10 points]

(d) Using graphical methods, show that the equation for a circular orbit that you found in part (b) has either one or three solutions, depending on whether $\ell^2/mRK$ is smaller or larger than a certain rational number that you will find. [10 points]

Problem 4. A homogeneous solid cube with mass $M$ and sides of length $a$ is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a tiny displacement and allowed to fall. (This is done in a uniform gravitational field with acceleration $g$. The moment of inertia of a cube about an axis through its center and parallel to an edge is $I = Ma^2/6$.)

(a) Find the angular velocity of the cube when one face strikes the plane, assuming that the edge cannot slide due to friction. [15 points]

(b) Same question as (a), but now assuming that the edge slides without friction on the plane. [15 points]

(c) For the frictionless case in part (b), what is the force exerted by the surface on the cube just before the face strikes the plane? [10 points]