Problem 1. A thin hollow cylinder of radius $R$ and mass $M$ slides across a rough horizontal surface with an initial linear velocity $V_0$. As it slides, it also has an initial angular velocity $\omega_0$ as shown in the figure. (Note that positive $\omega_0$ tends to produce rolling corresponding to motion in the direction opposite to $V_0$.) Let the coefficient of friction between the cylinder and the surface be $\mu$.

(a) How long will it take till the sliding stops? [16 points]
(b) What is the velocity of the center of mass of the cylinder at the time the sliding stops? [16 points]
(c) How do the results in (a) and (b) change for $\omega_0$ in the opposite direction from that shown in the figure? [8 points]

Problem 2. A block of mass $M$ is constrained to move without friction along a horizontal line (the $x$ axis in the figure). A simple pendulum of length $L$ and mass $m$ hangs from the center of the block. The pendulum moves in the $xy$ plane.

(a) [20 points] Find the Lagrangian and the equations of motion for the system.
(b) [20 points] Find the normal modes and normal frequencies of the system, assuming that the pendulum always makes a small angle with the vertical.
Problem 3. A uniform circular thin membrane with radius $R$ and mass/area $\mu$ is attached to a rigid support along its circumference, like a drumhead. Points on the membrane in the equilibrium position are labeled by polar coordinates $(r, \theta)$. The membrane has constant tension $C$, so that under a small displacement $f(r, \theta, t)$ from equilibrium, each area element $da$ contributes potential energy $\frac{1}{2}C(\nabla f)^2 da$.

(a) Find a wave equation satisfied by small time-dependent displacements $f(r, \theta, t)$ of the membrane from the equilibrium position. [16 points]

(b) Show that solutions can be found of the form $f = R(r)S(\theta)T(t)$, where:

$$\frac{d^2 S}{d\theta^2} + k^2 S = 0,$$
$$\frac{d^2 T}{dt^2} + n^2 T = 0,$$

where $k$ and $n$ are constants, and $R(r)$ satisfies a second-order ordinary differential equation that you will determine. [16 points]

(c) Briefly describe how you would determine the vibrational frequencies of the membrane. [8 points]

Problem 4. A particle of mass $m$ moves subject to a central potential:

$$V(r) = -\frac{k}{r^n}$$

where $k$ and $n$ are positive constants.

(a) If the particle has angular momentum $L$, what is the radius $R$ for which the orbit is circular? [10 points]

(b) Suppose the motion is close to the circular orbit mentioned in part (a). Writing $r(t) = R + \delta r(t)$ and assuming that $\delta r(t)$ is small, find an equation for $\delta r(t)$. [15 points]

(c) Solve this equation for $\delta r$ and discuss the stability of circular orbits, for different values of $n$. [15 points]