Problem 1. Object 1 (mass $m$) is attached to object 2 (mass $3m$) by a spring of unstretched length $L$ and spring constant $k$. As shown in the figure, the two masses are constrained to move on a circle with radius $R$. The spring is also constrained to be on the circle. Ignore gravity and friction.

(a) Find the kinetic energy of the system in terms of the coordinates $\theta_1$ and $\theta_2$. [6 points]
(b) Find the potential energy of the system in terms of the coordinates $\theta_1$ and $\theta_2$. [6 points]
(c) Find the Lagrangian of the system. [2 points]
(d) Find the two Lagrange equations of motion for coordinates $\theta_1$ and $\theta_2$. [9 points]
(e) Use the equations of motion to find the general solution for the motion of the two objects. [9 points]
(f) At time $t = 0$, both masses are at rest, $\theta_2 = 0$, and the spring is at twice its natural, unstretched length. Find the subsequent motion. [8 points]
Problem 2. A point particle of mass $m$ moves subject to a 3-dimensional central potential:

$$V(r) = -\frac{k}{r^n}$$

where $k$ and $n$ are positive constants.

(a) If the particle has angular momentum $L$, what is the radius $R$ for which the orbit is circular? [10 points]

(b) Suppose the motion is close to the circular orbit mentioned in part (a). Writing $r(t) = R + \delta r(t)$ and assuming that $\delta r(t)$ is small, find an equation of motion for $\delta r(t)$. Write your equation in a form that does not involve the angular momentum $L$. [15 points]

(c) Solve this equation for $\delta r$. For what values of $n$ are the circular orbits stable? [15 points]

Problem 3. A point particle of mass $m$ is fixed to the bottom end of a thin wire suspended from a fixed point on the ceiling. The thin wire has total mass $M$ and length $L$. The acceleration due to gravity is $g$. At time $t = 0$, the point $m$ is given a very small tap.

(a) Find the tension in the wire and the speed of waves in the wire as a function of $y$, the distance from $m$. [16 points]

(b) Find the total time needed for the perturbation to reach the top end of the wire (the ceiling). [24 points]

Problem 4. A uniform solid spherical ball of mass $M$ and radius $R$ rests on a horizontal surface. Assume a constant coefficient of friction $\mu$ (this means that the frictional force is equal to the normal force multiplied by $\mu$). The acceleration due to gravity is $g$. At time $t = 0$, the ball is struck impulsively on center, causing it to go instantaneously from rest to a horizontal speed $v_0$ with no initial rotation.

(a) Find the horizontal speed, and the angular velocity of the ball about its center, as a function of $t$. [16 points]

(b) Find the distance travelled by the ball until it begins to roll without slipping. [24 points]

[Hint: the moment of inertia of the sphere about its center is $\frac{2}{5}MR^2$.]