Problem 1. Consider a galaxy consisting of a very large number $N$ of identical stars, each of mass $m$, distributed uniformly as a very thin disk with radius $R$. The galaxy also has a concentric spherical halo of dark matter, with a finite but very large radius and uniform mass density $\rho$. The stars only interact gravitationally with the dark matter and each other.

(a) If the star orbits are perfectly circular, find the speed $v$ and the angular momentum $\ell$ (about the center of the galaxy) for each star, as a function of the star’s distance from the center $r$ and $N, m, R, \rho,$ and $G$ (Newton’s constant). [14 points]

(b) Now consider star orbits that can be non-circular. The equation of motion of $r$ for each star with angular momentum $\ell$ is the same as that of an equivalent 1-dimensional problem. For that problem, find the effective potential as a function of $r$. Your answer may depend on $N, m, R, \rho, G,$ and $\ell$. [13 points]

(c) Consider a star with a nearly circular orbit $r(t) = r_0 + \epsilon(t)$, where $r_0$ is constant and $\epsilon(t)$ is very small. Find an expression for the frequency of oscillations for $\epsilon$. [13 points]

Problem 2. A solid cube with side $2a$ has mass $M$ uniformly distributed throughout its volume. It rotates frictionlessly on a fixed horizontal axis which passes through the centers of two opposite sides. A point mass $m$ is attached to one corner of the cube. There is a constant downward gravitational field with acceleration $g$.

(a) Find the Lagrangian and the corresponding equation of motion for the system. [16 points]

(b) Find the frequency of small oscillations when the mass $m$ is near its lowest point. [10 points]

(c) If the mass $m$ is held at the same height as the axis of rotation, and then released, how much time will it take to reach its lowest point? You may leave your answer in terms of a definite integral over a single variable. [14 points]
Problem 3. Consider a system of two masses $m$ and three identical springs with spring constant $k$ between two stationary walls as shown. At equilibrium, the lengths of the springs are $a$, and if they were unstretched their lengths would be $b$. Consider only longitudinal motions (along the axis of the springs).

(a) Find the Lagrangian and Lagrange’s equations for the system. [14 points]
(b) Find the normal-mode frequencies of vibration and the eigenvectors. [13 points]
(c) Suppose that at time $t = 0$ the mass on the left is displaced from equilibrium by a distance $X$ to the right, while the mass on the right is not displaced, and both masses are at rest. Compute the motion of the left mass for $t > 0$. [13 points]

Problem 4. A speeding train car of mass $M$, moving with speed $v$, is to be stopped with a coiled-spring buffer of uncompressed length $\ell$ and spring constant $k$. If the spring becomes fully compressed, the spring constant will suddenly change to become very large. Assuming that you can choose $k$ to be any fixed constant that you want, what is the minimum value of $\ell$ needed to assure that the maximum absolute value of the deceleration of the train does not exceed $a_{\text{max}}$? What value of $k$ should you choose to achieve this? [40 points]