Problem 1. A particle of mass \( M \) is constrained to move on a smooth horizontal plane. A second particle of mass \( m \) is attached to it by hanging from a string passing through a hole in the plane as shown, and is constrained to move in a vertical line in a uniform gravitational field of acceleration \( g \). All motion is frictionless and the string is massless.

(a) Find the Lagrangian for the system and derive the equations of motion. [15 points]

(b) Consider solutions in which \( M \) moves in a circle with a constant speed \( v_0 \). Find the radius of the circle \( r_0 \) in terms of the other quantities. [12 points]

(c) Show that the solution in part (b) is stable and find the angular frequency of small oscillations about the stable circular orbit. [13 points]

Problem 2. Consider pointlike particles of mass \( m \) which approach a sphere of mass \( M \) and radius \( R \). The particles are attracted to the sphere in accordance with Newton’s law. When they are very far away, the particles have velocity \( v_\infty \). You may assume that \( m \ll M \). Find the effective cross-section (with units of area) for the particles to strike the sphere. [40 points]
Problem 3. Consider an infinite number of identical pendulums of mass $M$ in a uniform gravitational field with acceleration $g$, each hanging by a massless string of length $\ell$, and coupled to each other with massless springs of spring constant $K$ as shown. In the equilibrium position, the springs are at their natural length, $a$. The masses move only in the plane of the page, and with only a small displacement from equilibrium.

(a) Denote the small horizontal displacement of the $j$th mass from equilibrium as $u_j(t) = u(x, t)$, where $x = ja$ is the equilibrium position. Derive a wave equation of motion for $u(x, t)$ for this system as a second-order differential equation in $x$ and $t$, in the long wavelength approximation. [25 points]

(b) Find the dispersion relation (a relation between the angular frequency and the wavenumber). What is the minimum angular frequency? [15 points]
Problem 4. A lawn-mower engine contains a piston of mass $m$ that moves along $\dot{z}$ in a field of constant gravitational acceleration $\vec{g} = g\hat{z}$. The center of mass of the piston is connected to a flywheel of moment of inertia $I$ at a distance $R$ from its center by a rigid and massless rod of length $\ell$, as shown. The system has only one degree of freedom but two natural coordinates, $\phi$ and $z$.

(a) Express the Lagrangian in terms of $q_1 = z$, $q_2 = \phi$. [5 points]

(b) Write the constraint equation that connects the two coordinates. [5 points]

(c) From the above results, write down the two coupled equations of motion using the method of “undetermined multipliers”. Then eliminate the undetermined multiplier to obtain a single equation of motion (it can still involve both coordinates). [15 points]

(d) Find $p_\phi(z, \phi, \dot{\phi})$. [15 points]

Hint: A constraint equation of the form $C(q_i) = 0$ leads to Lagrange equation(s) of motion for the configuration variables $q_i$ that include additional terms $-\lambda \frac{\partial C}{\partial q_i}$, where $\lambda(q_i, t)$ is the undetermined multiplier.