NIU Physics PhD Candidacy Exam – Spring 2017 – Classical Mechanics

You may solve ALL FOUR problems, if you choose. Only the THREE BEST PROBLEM GRADES count towards your score.
Total points on each problem = 40. Total possible score = 120.

Problem 1. Three masses $m_1$, $m_2$, and $m_3$ are suspended as shown in a uniform gravitational field with acceleration $g$. Masses $m_1$ and $m_3$ are connected by an unstretchable string which moves over a massless pulley whose center does not move. Mass $m_2$ is suspended from mass $m_1$ by a spring with constant $k$ and unstretched length $L$. Call $x$ and $z$ the heights of masses $m_1$ and $m_2$, respectively, as shown.

(a) Find the Lagrangian of the system, using $x$ and $y = z - x - L$ as your configuration variables. [10 points]

(b) Find the equations of motion of the system. [10 points]

(c) Solve for the relative motion of the masses $m_1$ and $m_2$, and in particular find the angular frequency of oscillation [10 points]

(d) What is the oscillation frequency if $m_1 \gg m_2, m_3$? Show using your answer above and/or physical arguments. [5 points]

(e) What is the oscillation frequency if $m_2 \gg m_1, m_3$? Show using your answer above and/or physical arguments. [5 points]

Problem 2. A point particle of mass $M_1$ approaches a massive sphere of radius $R$ and mass $M_2$. The two masses attract each other according to Newton’s law of gravitation. When very far away from each other, $M_1$ has initial speed $v_0$ while the sphere $M_2$ is initially at rest.

(a) If the initial impact parameter distance is $b$, find the distance of closest approach of the centers of the two objects. [20 points]

(b) Find the cross-section for the two objects to collide. [12 points]

(c) Give your answers and physical explanations for the limits $v_0 \to 0$ and $v_0 \to \infty$. [8 points]
Problem 3. A thin rod of length $\ell$ and mass $M$ is supported at one end by a smooth floor on which it slides without friction. The rod falls in a uniform gravitational field with acceleration $g$, starting from rest with an initial angle $\theta_0$ relative to the horizontal, as shown.

(a) Find the Lagrangian for the system. \([10 \text{ points}]\]
(b) Find the Hamiltonian for the system. \([6 \text{ points}]\]
(c) Find the time needed for the rod to fall to the floor. You may leave your answer in terms of a definite integral. \([18 \text{ points}]\]
(d) How far does the lower end of the rod move during this time? \([6 \text{ points}]\]

Problem 4. A water drop falls vertically in a uniform gravitational field $g$, growing by accretion of much smaller micro-droplets of water in the atmosphere. The micro-droplets have negligible velocity and air resistance is neglected. As the falling drop grows, it maintains a perfectly spherical shape, with radius we will call $R$. The mass density of water is a constant $\rho$, and the density of the micro-droplets (per volume of atmosphere) is $\epsilon \rho$, so that the mass of the falling drop increases at a rate given by its speed multiplied by $(\epsilon \rho)(\pi R^2)$.

(a) Find a relation between the speed of the falling water drop and the time derivative of its size, and use it to show that $R(t)$ obeys the differential equation

$$\frac{d^2 R}{dt^2} + \frac{n}{R} \left( \frac{dR}{dt} \right)^2 = C,$$

where $n$ is a certain integer that you will determine, and $C$ is a constant that you will find in terms of the given quantities. \([22 \text{ points}]\]
(b) Find a general relation for the downward acceleration of the drop in terms of $g$, $\epsilon$, and the instantaneous values of its speed $v$ and radius $R$. \([8 \text{ points}]\]
(c) Starting from the results found in part (a), try a power law solution of the form $R(t) = At^\alpha$, and solve for the constants $A$ and $\alpha$, to show that for large $t$ the downward acceleration of the drop approaches the constant $g/N$, where $N$ is another integer that you will determine. \([12 \text{ points}]\]
Problem 1. Object 1 (mass $m$) is attached to object 2 (mass $3m$) by a spring of unstretched length $L$ and spring constant $k$. As shown in the figure, the two masses are constrained to move on a circle with radius $R$. The spring is also constrained to be on the circle. Ignore gravity and friction.

(a) Find the kinetic energy of the system in terms of the coordinates $\theta_1$ and $\theta_2$. [6 points]
(b) Find the potential energy of the system in terms of the coordinates $\theta_1$ and $\theta_2$. [6 points]
(c) Find the Lagrangian of the system. [2 points]
(d) Find the two Lagrange equations of motion for coordinates $\theta_1$ and $\theta_2$. [9 points]
(e) Use the equations of motion to find the general solution for the motion of the two objects. [9 points]
(f) At time $t = 0$, both masses are at rest, $\theta_2 = 0$, and the spring is at twice its natural, unstretched length. Find the subsequent motion. [8 points]
Problem 2. A point particle of mass $m$ moves subject to a 3-dimensional central potential:

$$V(r) = -\frac{k}{r^n}$$

where $k$ and $n$ are positive constants.

(a) If the particle has angular momentum $L$, what is the radius $R$ for which the orbit is circular? [10 points]

(b) Suppose the motion is close to the circular orbit mentioned in part (a). Writing $r(t) = R + \delta r(t)$ and assuming that $\delta r(t)$ is small, find an equation of motion for $\delta r(t)$. Write your equation in a form that does not involve the angular momentum $L$. [15 points]

(c) Solve this equation for $\delta r$. For what values of $n$ are the circular orbits stable? [15 points]

Problem 3. A point particle of mass $m$ is fixed to the bottom end of a thin wire suspended from a fixed point on the ceiling. The thin wire has total mass $M$ and length $L$. The acceleration due to gravity is $g$. At time $t = 0$, the point $m$ is given a very small tap.

(a) Find the tension in the wire and the speed of waves in the wire as a function of $y$, the distance from $m$. [16 points]

(b) Find the total time needed for the perturbation to reach the top end of the wire (the ceiling). [24 points]

Problem 4. A uniform solid spherical ball of mass $M$ and radius $R$ rests on a horizontal surface. Assume a constant coefficient of friction $\mu$ (this means that the frictional force is equal to the normal force multiplied by $\mu$). The acceleration due to gravity is $g$. At time $t = 0$, the ball is struck impulsively on center, causing it to go instantaneously from rest to a horizontal speed $v_0$ with no initial rotation.

(a) Find the horizontal speed, and the angular velocity of the ball about its center, as a function of $t$. [16 points]

(b) Find the distance travelled by the ball until it begins to roll without slipping. [24 points]

[Hint: the moment of inertia of the sphere about its center is $\frac{2}{5}MR^2$.]
Problem 1. Two masses $m_1$ and $m_2$ are attached via a massless spring of natural (unstretched) length $L$ and spring constant $k$, as shown in the figure. The masses are freely falling in a uniform gravitational field with acceleration $g$. The mass $m_1$ is located vertically above the mass $m_2$, with heights $x$ and $y$ respectively, and all motion is vertical. At time $t=0$, mass $m_1$ is at height $x_0$ and traveling vertically downward with velocity $v_0$, and mass $m_2$ is at height $y_0$ and is at rest. The masses are initially a distance $L$ apart ($x_0 - y_0 = L$), so that the spring is neither compressed nor stretched.

(a) Write down the Lagrangian for the system in terms of $x$ and $y$. [8 points]

(b) Redefine the coordinates in terms of the center of mass of the system and the relative coordinate between the masses. [8 points]

(c) Rewrite the Lagrangian in terms of the new coordinates and find the equations of motion. [8 points]

(d) Solve the equations of motion for the new coordinates for all times $t > 0$. [8 points]

(e) Rewrite the solution of the equations of motion in terms of the original coordinates $x$ and $y$ for all times $t > 0$. [8 points]
Problem 2. A massless string of length $L$ passes through a hole in a horizontal table. A point pass $M$ at one end of the string moves frictionlessly on the table (i.e. with two degrees of freedom), and another point mass $m$ hangs vertically from the other end. The system is in a uniform gravitational field with acceleration $g$.

![Diagram of a mass $M$ on a string and a hanging mass $m$.]

(a) Write the Lagrangian for the system. [13 points]

(b) Suppose the mass $M$ on the table initially has a speed $v$. Under what conditions will the hanging mass remain stationary? [13 points]

(c) Starting from the situation in part (b), the hanging mass is pulled down very slightly and then released. Compute the angular frequency of the subsequent oscillatory motion of the hanging mass. [14 points]

Problem 3. A table consists of a horizontal thin uniform circular disk of radius $R$ and mass $M$, supported on its rim by three thin vertical legs that are equidistant from each other. The acceleration due to gravity is $g$.

(a) Suppose two of the legs are suddenly removed. Find the force exerted by the third leg on the table top immediately after. [20 points]

(b) Suppose instead that just one of the legs is suddenly removed. Find the force exerted by each of the remaining two legs immediately after. [20 points]

Possibly useful information: for a thin solid disk of mass $M$ and radius $R$, the three principal moments of inertia about its center are: $\frac{1}{4}MR^2$, $\frac{1}{4}MR^2$, and $\frac{1}{2}MR^2$. 
Problem 4. A uniform string of length $L$ undergoes small transverse oscillations. The tension of the string is $T$ and the mass per unit length is $\mu$. The equilibrium position of the string lies along the $x$ axis and the transverse displacement at time $t$ is restricted to the $y$ direction and is denoted by $y(x, t)$. One end of the string, at $x = 0$, is fixed, so that $y(0, t) = 0$ for all $t$. The other end of the string is attached to a point particle of mass $m$ that is free to move frictionlessly in the $y$ direction at fixed $x = L$. Neglect gravity.

(a) Write down the wave equation for small amplitude displacements $y(x, t)$, and express the velocity of propagation of transverse waves in terms of $T$ and $\mu$. (For this part only, you may obtain the answers just by using dimensional analysis to supplement your memory, if you wish.) [6 points]

(b) Show that the boundary condition for the mass $m$ at $x = L$ for small displacements has the form:

$$-C \frac{\partial y}{\partial x} = \frac{\partial^2 y}{\partial t^2}, \quad \text{(at} \ x = L\text{)}$$

where $C$ is a constant that you will determine in terms of the given quantities. [12 points]

(c) Use the boundary condition above to obtain a transcendental equation that implicitly determines the wavenumbers $k$ of the normal modes of the system. Show graphically that there are an infinite number of solutions. [16 points]

(d) For each of the extreme cases $m = 0$ and $m = \infty$, use your answer to part (c) to solve for the allowed wavenumbers $k$ for the normal modes. [6 points]
Problem 1. Consider a circular disc of radius $R$ as shown above. The disc is forced to rotate counterclockwise about its center at a constant angular velocity $\omega$. A pendulum with mass $m$ and length $L$ hangs from a point $P$ on the edge of the disc. There is a uniform gravitational field with acceleration $g$ pointing down. At time $t = 0$, the point $P$ is at its highest point.

(a) Find the position and velocity of the point $P$ as a function of time. (4 points)

(b) Find the Lagrangian for the system in terms of the configuration variable $\phi$, which is the angle made by the pendulum with the vertical. Simplify the result so that each term has at most one trigonometric function. [The following identities might be useful: $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and $\cos(A + B) = \cos A \cos B - \sin A \sin B$.] (14 points)

(c) Find the equation of motion for $\phi$. (12 points)

(d) If the pendulum mass $m$ is instantaneously at rest directly below the center of the disc (and $P$) at time $t = 0$, find the subsequent motion of $\phi$ for small $t$, as an expansion up to and including terms of order $t^3$. [Hints for part (d): try a solution of the form $\phi = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$.]

To find the correct result, sine functions need only be expanded to linear order in their arguments.] (10 points)
Problem 2  A simple pendulum (string length $L$ and mass $m$) is freely swinging in a plane under the gravitational force, $mg$, directed downward, as shown above.

(a) Find the equation of motion of this pendulum. Do not assume that $\theta$ is small. [4 points]

(b) Show that this non-linear differential equation can be approximated by a linear differential equation (simple harmonic oscillator) under a certain condition. What is the condition, and what is the period $T_0$ of this “approximated” simple harmonic oscillator? [4 points]

(c) For the non-linear differential equation in part (a), suppose that the motion is such that the maximum angular displacement is $\theta = \theta_0$. Find the angular velocity $d\theta/dt$ as a function of $\theta$. Write your answer in terms of $g$, $L$, $\theta_0$, and $\theta$. [16 points]

(d) Now suppose that $\theta = 0$ at time $t = 0$. Note that $\theta = \theta_0$ (the maximum angular deviation) at time $t = T/4$, where $T$ is the period of the non-linear oscillation. Find an expression for $T$ in terms of $\theta_0$ and $T_0$ [the period found in part (b)], in the form of an expansion in $\sin(\theta_0/2)$. Keep terms up to and including second order in that expansion. [16 points]

[Hints for part (d): You may find it useful to replace cosines using the formula $\cos(\theta) = 1 - 2\sin^2(\theta/2)$. You may then find it useful to write $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$, which defines a new variable $\phi$. Finally, the expansion $1/\sqrt{1-x^2} = 1 + x^2/2 + \ldots$ may be useful.]
Problem 3 Two identical thin uniform rods, each of length \(d\) and mass \(m\), are hinged together at the point \(A\). The rod on the left has one end hinged at the fixed point \(O\), while the end \(B\) of the other rod slides freely along the horizontal \(x\) axis. The system is in a uniform vertical gravitational field with acceleration \(g\). All motion is frictionless.

(a) Find the total kinetic energy of the system. You should find the result:

\[ T = m d^2 \dot{\phi}^2 (a + b \sin \phi + c \sin^2 \phi) \]

where \(a\), \(b\) and \(c\) are constant numbers that you will determine. (Exactly one of \(a\), \(b\), and \(c\) is 0.) [25 points]

(b) Suppose the system is at rest at time \(t = 0\) with \(\phi = \phi_0\). What is the velocity of the hinge \(A\) when it hits the horizontal \(x\) axis? [15 points]

Problem 4. Consider a particle of mass \(m\) scattering from a central potential \(V = -k/r^n\), where \(k\) and \(n\) are positive constants. The particle approaches from very far away with a non-zero impact parameter \(b\) and initial velocity \(v_0\).

(a) Show that for the particle to have a chance at hitting the origin \((r = 0)\), it is necessary that \(n\) is greater than or equal to a certain number that you will determine. [15 points]

(b) Now taking \(n = 4\), show that a necessary and sufficient condition for the particle to hit the origin is that \(b < b_{\text{crit}}\), where \(b_{\text{crit}}\) is a quantity that you will determine in terms of \(k\), \(v_0\), and \(m\). [20 points]

(c) Still taking \(n = 4\), what is the cross section for particles to hit the origin? [5 points]
Problem 1. Consider a stationary inclined plane ramp with a fixed angle $\theta$, with three blocks with equal masses $M$ attached by a spring of constant $k$ and a string over a fixed pulley at the top of the ramp, as shown below. The positions of the blocks are described by the distances $x(t)$ (the distance from the pulley to the higher block on the ramp) and $y(t)$ (the extension of the spring), as shown. The system is in a uniform gravitational field of acceleration $g$. The unstretched length of the spring is 0, and the pulley is massless.

(a) [12 points] Find the Lagrangian and the equations of motion of the system.

(b) [20 points] The two blocks on the ramp are released from rest at the top with $x = y = 0$ at time $t = 0$. Find the motions of the blocks as a function of time.

(c) [8 points] The choice $\theta = 30$ degrees has a special significance, which should be apparent in the results of part (b). What is it?

Problem 2. A bead of mass $m$ moves without friction along a fixed horizontal line. A thin rod of length $a$ and total mass $M$ (uniformly distributed along its length) hangs from the bead, attached by a very short massless string. The rod is constrained to swing in the plane of the page. The system is in a uniform gravitational field with acceleration $g$ downwards.

(a) [20 points] Find the Lagrangian of the system and the equations of motion.

(b) [20 points] Find the normal modes and corresponding frequencies of small oscillations.
Problem 3. Consider a rocket with total initial mass $m$. Exactly half of $m$ is fuel, which is expelled from the rocket at a constant rate $k$ (mass per unit time) and with speed $u$ with respect to the rocket. There is a uniform gravitational field $g$ downwards, and the motion of the rocket is entirely vertical. Let the height of the rocket as a function of time be $x(t)$. The rocket is at rest on the ground (at $x = 0$) and begins expelling fuel at time $t = 0$.

(a) [16 points] Find a differential equation satisfied by the height of the rocket.

(b) [6 points] Find the condition on $m, k, u, g$ for the rocket to lift off immediately at $t = 0$, and the condition for the rocket to lift off at all (at any $t$).

(c) [18 points] Assuming that the rocket takes off immediately at $t = 0$, find expressions for the height and speed of the rocket at the moment when it runs out of fuel. Write your answers as algebraic expressions in terms of only the quantities $m, k, u, g$.

The following indefinite integral may or may not be useful:

\[
\int \ln(1 - az) \, dz = \left( z - \frac{1}{a} \right) \ln(1 - az) - z.
\]

Problem 4. The force of attraction between a star of mass $M$ and a planet of mass $m$ (where $m \ll M$) is:

\[
F = \frac{a}{r^2} + \frac{3b\ell^2}{r^4}
\]

where $\ell$ is the angular momentum of the planet and $a, b$ are both positive constants. [Note: this does approximate the force of attraction between a planet and a black hole, in the non-relativistic limit, with $a = GMm$.]

(a) [15 points] Under what conditions is a stable circular orbit possible? Give the radius of the stable circular orbit in terms of the given parameters $(M, m, a, b, \ell)$.

(b) [15 points] What is the smallest radius possible for any circular orbit as a function of $a$ and $b$, allowing for arbitrary $\ell$? (Hint: this occurs in the limit of very large $\ell$.) Is this circular orbit stable or unstable?

(c) [10 points] If the planet travels in a slightly non-circular orbit about a stable radius, find an expression for the angular frequency of small radial oscillations.
Problem 1. Consider a galaxy consisting of a very large number $N$ of identical stars, each of mass $m$, distributed uniformly as a very thin disk with radius $R$. The galaxy also has a concentric spherical halo of dark matter, with a finite but very large radius and uniform mass density $\rho$. The stars only interact gravitationally with the dark matter and each other.

(a) If the star orbits are perfectly circular, find the speed $v$ and the angular momentum $\ell$ (about the center of the galaxy) for each star, as a function of the star’s distance from the center $r$ and $N, m, R, \rho$, and $G$ (Newton’s constant). [14 points]

(b) Now consider star orbits that can be non-circular. The equation of motion of $r$ for each star with angular momentum $\ell$ is the same as that of an equivalent 1-dimensional problem. For that problem, find the effective potential as a function of $r$. Your answer may depend on $N, m, R, \rho, G$, and $\ell$. [13 points]

(c) Consider a star with a nearly circular orbit $r(t) = r_0 + \epsilon(t)$, where $r_0$ is constant and $\epsilon(t)$ is very small. Find an expression for the frequency of oscillations for $\epsilon$. [13 points]

Problem 2. A solid cube with side $2a$ has mass $M$ uniformly distributed throughout its volume. It rotates frictionlessly on a fixed horizontal axis which passes through the centers of two opposite sides. A point mass $m$ is attached to one corner of the cube. There is a constant downward gravitational field with acceleration $g$.

(a) Find the Lagrangian and the corresponding equation of motion for the system. [16 points]

(b) Find the frequency of small oscillations when the mass $m$ is near its lowest point. [10 points]

(c) If the mass $m$ is held at the same height as the axis of rotation, and then released, how much time will it take to reach its lowest point? You may leave your answer in terms of a definite integral over a single variable. [14 points]
Problem 3. Consider a system of two masses $m$ and three identical springs with spring constant $k$ between two stationary walls as shown. At equilibrium, the lengths of the springs are $a$, and if they were unstretched their lengths would be $b$. Consider only longitudinal motions (along the axis of the springs).

(a) Find the Lagrangian and Lagrange’s equations for the system. [14 points]
(b) Find the normal-mode frequencies of vibration and the eigenvectors. [13 points]
(c) Suppose that at time $t = 0$ the mass on the left is displaced from equilibrium by a distance $X$ to the right, while the mass on the right is not displaced, and both masses are at rest. Compute the motion of the left mass for $t > 0$. [13 points]

Problem 4. A speeding train car of mass $M$, moving with speed $v$, is to be stopped with a coiled-spring buffer of uncompressed length $\ell$ and spring constant $k$. If the spring becomes fully compressed, the spring constant will suddenly change to become very large. Assuming that you can choose $k$ to be any fixed constant that you want, what is the minimum value of $\ell$ needed to assure that the maximum absolute value of the deceleration of the train does not exceed $a_{\text{max}}$? What value of $k$ should you choose to achieve this? [40 points]
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Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Consider three identical springs with spring constant $\kappa$ and unstretched length $L$, connecting three identical point masses $\mu$. These masses and springs are constrained to move frictionlessly on a circle of radius $a$ as in the figure below. There is no gravity. The positions of the masses are described by three angles $\theta_1$, $\theta_2$, $\theta_3$ as shown.

(a) What is the kinetic energy of the system? [10 points]
(b) What is the potential energy of the system? Express it using a matrix. [10 points]
(c) Find the normal modes and frequencies of oscillation for the system. [20 points]

Problem 2. A small satellite with mass $m$ is in a circular orbit of radius $r$ around a planet of mass $M$. The planet’s thin atmosphere results in a frictional force $F = Av^\alpha$ on the satellite, where $v$ is the speed of the satellite and $A$ and $\alpha$ are constants. It is observed that with the passage of time, the orbit of the satellite remains circular, but with a radius that decreases very slowly with time according to $dr/dt = -C$, where $C$ is a constant independent of the orbit’s radius and the speed. Assume that $m \ll M$.

(a) Show that $\alpha$ can only be a certain integer, and find that integer. [20 points]
(b) Solve for the constant $A$. Express your answer in terms of $C$, the masses $m$ and $M$, and Newton’s gravitational constant $G$. [20 points]
Problem 3. Consider a point mass $m$ moving in the $(x, z)$ plane on the parabola $z = x^2/(2a)$ with $a > 0$ and subject to the constant gravitational field $g = (0, -g)$.

(a) Are there conserved quantities? If so, what are they? [8 points]
(b) Give the kinetic and potential energies for the point mass. What is the Lagrangian, if one chooses $x$ as the generalized coordinate? [8 points]
(c) Derive the canonical momentum $p$ conjugate to $x$ as well as the Lagrange equation for $x(t)$. [8 points]
(d) Determine the Hamiltonian function $H$. [8 points]
(e) Assuming $x \ll a$, obtain for $H$ a quadratic form in $x$ and $p$. Write down the canonical equations of motion for this approximate Hamiltonian function, and solve them assuming the initial condition $x(0) = 0, p(0) = p_0$. [8 points]

Problem 4. A uniform solid sphere of radius $r$ and mass $m$ rolls off of a fixed cylinder of radius $R$, starting nearly from rest at the top of the cylinder, in the Earth’s constant gravitational field with acceleration $g$. The moment of inertia of a solid sphere of radius $r$ and mass $m$ about its center is $\frac{2}{5}mr^2$.

(a) Write the Lagrangian for the system using the single configuration variable $\theta$ shown. [15 points]
(b) At what angle will the sphere leave the cylinder? [20 points]
(c) If the sphere had the same radius and total mass, but was a thin shell instead of a uniform solid, would the angle be greater or less than in part (b)? Explain your answer. [5 points]
Problem 1. Consider a circular hoop with mass $M$ and radius $R$ rolling down a fixed inclined plane without slipping. The inclined plane of length $L$ is placed in Earth’s gravitational field at a fixed angle $\Phi$ with respect to the horizontal plane. Use the generalized coordinates $x$ (measured along the inclined plane) and $\theta$ (the rotation angle of the hoop) to describe the motion of the hoop. At time $t = 0$, the hoop starts at rest with $x = 0$, $\theta = 0$.

(a) Draw a figure showing all the forces acting on the hoop. [4 points]
(b) Find the Lagrangian in terms of generalized coordinates and generalized velocities. [6 points]
(c) Determine the holonomic constraint $f(x, \theta) = 0$ between $x$ and $\theta$. [2 points]
(d) Eliminate one of the generalized coordinates in the Lagrangian by using $f(x, \theta) = 0$. [2 points]
(e) Find an equation of motion, and solve it to determine the motion of the hoop by finding $x(t)$ and $\theta(t)$. [6 points]
(f) By using the method of undetermined Lagrange multipliers with the holonomic constraint $f(x, \theta)$ find equations of motion, solve them in terms of $x(t)$ and $\theta(t)$, and find the force of constraint. [12 points]
(g) Determine the non-holonomic (semi-holonomic) constraint between generalized velocities $\dot{x}$ and $\dot{\theta}$, of the form $f(\dot{x}, \dot{\theta}) = 0$. [2 points]
(h) Use the method of undetermined Lagrange multipliers with the semi-holonomic constraint $f(\dot{x}, \dot{\theta})$ to find equations of motion, solve them in terms of $x(t)$ and $\theta(t)$, and find the force of constraint. [6 points]
Problem 2. Consider a point particle of mass $m$ moving in three dimensions in a central potential

$$V(r) = -\frac{\alpha}{r} - \frac{\beta}{r^2}$$

where $\alpha$ and $\beta$ are positive constants and $r$ is the distance from the origin. The particle approaches from very far away with speed $v$ and impact parameter $b$, as shown. (The dashed line is what the path would be if there were no potential.)

(a) Find $r_{\text{min}}$, the distance of closest approach of the particle to $r = 0$. Show that the particle will go through the origin if $\beta > \beta_c$, where $\beta_c$ is a critical value that you will determine in terms of the other given quantities. [12 points]

In the remainder of this problem, you should assume $\beta < \beta_c$, and you may leave your answers in terms of $r_{\text{min}}$ and the other given quantities.

(b) What is the maximum speed reached by the particle on its trajectory? [8 points]

(c) What is the maximum acceleration reached by the particle on its trajectory? [8 points]

(d) When the particle is very far from the origin again, find the angle by which it has been scattered from its original direction. You may leave your answer in terms of a single definite integral. [12 points]
Problem 3. A pendulum of fixed length $a$ is constrained to move in a vertical plane. This plane is forced to rotate at fixed angular velocity, $\Omega$, about a vertical axis passing through the pendulum’s pivot. The mass of the pendulum bob is $m$ and there is a uniform gravitational field with acceleration $g$ pointing down.

(a) Find all of the equilibrium points of the pendulum, and find whether they are stable or unstable. [25 points]
(b) Find the frequency of small oscillations about the point or points of stable equilibrium. [15 points]

Some possibly useful identities:

\[
\begin{align*}
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\sin(\cos^{-1}(x)) &= \sqrt{1 - x^2} \\
\cos(\sin^{-1}(x)) &= \sqrt{1 - x^2}
\end{align*}
\]
Problem 4. A thin straight rod of fixed length $2\ell$ and linear mass density $\rho$ is constrained to move with its ends on a circle of radius $R$, where $R > \ell$. The whole circle is held absolutely fixed in a vertical plane here on the Earth’s surface, and the contacts between the circle and the rod are frictionless.

(a) Find the Lagrangian for the motion of the rod. [18 points]

(b) Obtain the equations of motion for the rod. [10 points]

(c) Find the frequency of small oscillations, assuming that the departures from equilibrium are small. [12 points]
Problem 1. Consider a homogeneous cube with edge length \( a \) and mass \( m \). We study the rotational motion (angular velocity \( \omega \)) of the cube about different axes of rotation.

(a) Evaluate the inertia tensor assuming that the origin of the coordinate system is at the center of the cube. [10 points]

(b) Evaluate the kinetic energy for a rotation with angular velocity \( \omega \) about the \( x \) axis of the coordinate system, i.e., \( \vec{\omega} = (\omega, 0, 0) \). [10 points]

(c) Does the kinetic energy change as compared with (b) if the cube rotates about the axis \( A \) defined by \( x = y \) with \( z = 0 \) (see figure)? Justify your answer. [10 points]

(d) Evaluate the kinetic energy for a rotation about the axis \( B \), defined by \(-x = y = a/2\). [10 points]
Problem 2. A very long, perfectly flexible material is rolled up on a fixed, horizontal, massless, very thin axle. The rolled-up portion rotates freely on the axle. The material has a (small) thickness $s$, width $w$, and, when completely unrolled, length $\ell$. The mass density per unit volume of the material is $\rho$. At time $t = 0$, a length $x_0$ hangs from the roll and the system is at rest. For $t > 0$, the material unrolls under the influence of a constant gravitational field $g$ (downward). [Useful information: the moment of inertia of a solid cylinder of mass $M$ and radius $R$ about its axis of symmetry is $I = MR^2/2$.]

(a) Using the length $x$ of the hanging part of the material as your configuration variable, find the Lagrangian of the system. [15 points]
(b) Find the canonical momentum conjugate to $x$, and the Hamiltonian of the system. [9 points]
(c) What is the velocity of the free end of the material at the time when half of it is off the roll ($x = \ell/2$)? [16 points]
Problem 3. A small bead of mass $M$ is free to slide on a frictionless, uniform wire, also of mass $M$, which is formed into a circle of radius $R$. The circular wire is suspended at one point from a fixed pivot, so that it is free to swing under gravity, but only within its own plane (the plane of the page in the figure below). The gravitational field is a constant $g$ (downward), and the small triangle in the figure represents the pivot point.

(a) Find the Lagrangian of the system using appropriate configuration variables, and the equations of motion. [20 points]
(b) Find the angular frequencies of small oscillation modes. [20 points]

Problem 4. Consider the (planar) motion of a particle of mass $m$ and initial angular momentum $L$ in the central 3-dimensional potential corresponding to a “spring” with a non-zero relaxed length $a$:

$$V(r) = \frac{1}{2}k(r - a)^2$$

(a) [20 points] Find a condition relating the radius $r_0$ of circular orbits and the given quantities. (It is not necessary to solve for $r_0$.)
(b) [20 points] For nearly circular orbits, find an expression for the period of time between successive radial maxima for $a/r_0 \ll 1$, expressing your answer to first order in $a/r_0$ (with $L$ eliminated from the answer). Use this to find the angular change between successive radial maxima in the limit $a/r_0 \rightarrow 0$, and check that the orbits are closed in that limit.
Problem 1. A thin rigid uniform bar of mass \( M \) and length \( L \) is supported in equilibrium in a horizontal position by two massless springs attached to each end, as shown. The springs have the same force constant \( k \), and are suspended from a horizontal ceiling in a uniform gravitational field. The motion of the bar is constrained to the \( x, z \) plane, and the center of gravity of the bar is further constrained to move parallel to the vertical \( x \)-axis.

(a) Write a Lagrangian describing the dynamics of this system for small deviations from equilibrium, using as configuration variables the heights of the endpoints of the bar. 
[15 points]
(b) Find the normal modes and angular frequencies of small vibrations of the system. 
[25 points]

Problem 2. A particle of mass \( m \) moves in one dimension in a harmonic oscillator potential, and is also subject to a time-dependent force \( F(t) \) so that the Lagrangian is:

\[
L = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2 + xF(t).
\]

Let

\[
F(t) = \begin{cases} 
0 & \text{(for } t \leq 0), \\
\frac{4}{T}F_0 & \text{(for } 0 < t < T), \\
F_0 & \text{(for } t \geq T), 
\end{cases}
\]

Suppose that the particle is at rest in equilibrium at \( t = 0 \).

(a) Derive the equation of motion and find the solution for \( x(t) \) when \( 0 < t < T \). 
[20 points]
(b) Find the solution for \( x(t) \) for all times \( t > T \). 
[20 points]
Problem 3. A slowly moving spinning planet of radius $R$, with no atmosphere, encounters a region with many tiny meteors moving in random directions. A thin layer of dust of thickness $h$ then forms from the fall of meteors hitting the planet from all directions. Let the original uniform density of the planet be $\rho$, and the density of the layer of meteor dust be $\rho_d$. Assuming that $h \ll R$, show that the fractional change in the length of the day is

$$N_1 \left( \frac{h}{R} \right)^{N_2} \left( \frac{\rho_d}{\rho} \right)^{N_3},$$

where $N_1$, $N_2$, and $N_3$ are each certain non-zero integers that you will determine. (The moment of inertia of a sphere about an axis through its center is $\frac{2}{5} MR^2$.) [40 points]

Problem 4. Express your answers for the questions below in terms of the gravitational constant $G$, the masses, and the distance $L$. Assume that the stars move on circular orbits.

(a) What is the rotational period $T$ of an equal-mass ($M_1 = M_2 = M$) double star of separation $L$? [10 points]

(b) What is the rotational period $T$ of an unequal-mass ($M_1 \neq M_2$) double star of separation $L$? [15 points]

(c) What is the rotational period $T$ of an equal-mass ($M_1 = M_2 = M_3 = M$) triple star, where the stars are located at the corners of an equilateral triangle (side $L$)? [15 points]
NIU Physics PhD Candidacy Exam – Spring 2012 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. A coupled oscillator consists of two springs with equal spring constants $k$ and two equal masses $m$, which hang from a fixed ceiling in a uniform gravitational field with acceleration $g$. The system oscillates in the vertical direction only.

(a) Find the Lagrangian for the system. [10 points]
(b) Find the angular frequencies of the normal modes of oscillation. [20 points]
(c) In the slower mode, find the ratio of the amplitude of oscillation of the upper mass to that of the lower mass. [10 points]

Problem 2. A particle of mass $m$ moves along the $x$-axis under the influence of a time-dependent force given by $F(x,t) = -kx e^{-t/\tau}$, where $k$ and $\tau$ are positive constants.

(a) Compute the Lagrangian function. [7 points]
(b) Find the Lagrangian equation of motion explicitly. [7 points]
(c) Compute the Hamiltonian function in terms of the generalized coordinate and generalized momentum. (Show clearly how you got this.) [7 points]
(d) Determine Hamilton’s equations of motion explicitly for this particular problem (not just general formulae). [7 points]
(e) Does the Hamiltonian equal the total energy of the mass? Explain. [6 points]
(f) Is the total energy of the mass conserved? Explain. [6 points]
Problem 3. Consider a particle of mass $m$ moving in a three-dimensional central potential

$$V(r) = -\frac{K}{r - R},$$

where $K$ and $R$ are fixed positive constants. Let the angular momentum of the particle be $\ell$.

(a) Find the differential equation of motion for $r$, the radial coordinate of the particle. [10 points]

(b) Derive an algebraic equation for the possible values of the radius $r_c$ of a circular orbit. (You do not need to solve the equation.) [10 points]

(c) Consider a small perturbation of a circular orbit, $r = r_c + \epsilon$. Find a linearized differential equation for $\epsilon$, written in terms of $r_c$, $R$, and $K$ (with $\ell$ eliminated). Use it to show that circular orbits are stable if either $r_c < R$ or $r_c > nR$, where $n$ is a certain integer that you will find. [10 points]

(d) Using graphical methods, show that the equation for a circular orbit that you found in part (b) has either one or three solutions, depending on whether $\ell^2/mRK$ is smaller or larger than a certain rational number that you will find. [10 points]

Problem 4. A homogeneous solid cube with mass $M$ and sides of length $a$ is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a tiny displacement and allowed to fall. (This is done in a uniform gravitational field with acceleration $g$. The moment of inertia of a cube about an axis through its center and parallel to an edge is $I = Ma^2/6$.)

(a) Find the angular velocity of the cube when one face strikes the plane, assuming that the edge cannot slide due to friction. [15 points]

(b) Same question as (a), but now assuming that the edge slides without friction on the plane. [15 points]

(c) For the frictionless case in part (b), what is the force exerted by the surface on the cube just before the face strikes the plane? [10 points]
Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. A particle of mass $M$ is constrained to move on a smooth horizontal plane. A second particle of mass $m$ is attached to it by hanging from a string passing through a hole in the plane as shown, and is constrained to move in a vertical line in a uniform gravitational field of acceleration $g$. All motion is frictionless and the string is massless.

(a) Find the Lagrangian for the system and derive the equations of motion. [15 points]

(b) Consider solutions in which $M$ moves in a circle with a constant speed $v_0$. Find the radius of the circle $r_0$ in terms of the other quantities. [12 points]

(c) Show that the solution in part (b) is stable and find the angular frequency of small oscillations about the stable circular orbit. [13 points]

Problem 2. Consider pointlike particles of mass $m$ which approach a sphere of mass $M$ and radius $R$. The particles are attracted to the sphere in accordance with Newton’s law. When they are very far away, the particles have velocity $v_\infty$. You may assume that $m \ll M$. Find the effective cross-section (with units of area) for the particles to strike the sphere. [40 points]
Problem 3. Consider an infinite number of identical pendulums of mass $M$ in a uniform gravitational field with acceleration $g$, each hanging by a massless string of length $\ell$, and coupled to each other with massless springs of spring constant $K$ as shown. In the equilibrium position, the springs are at their natural length, $a$. The masses move only in the plane of the page, and with only a small displacement from equilibrium.

(a) Denote the small horizontal displacement of the $j$th mass from equilibrium as $u_j(t) = u(x,t)$, where $x = ja$ is the equilibrium position. Derive a wave equation of motion for $u(x,t)$ for this system as a second-order differential equation in $x$ and $t$, in the long wavelength approximation. [25 points]

(b) Find the dispersion relation (a relation between the angular frequency and the wavenumber). What is the minimum angular frequency? [15 points]
Problem 4. A lawn-mower engine contains a piston of mass $m$ that moves along $\hat{z}$ in a field of constant gravitational acceleration $\vec{g} = g\hat{z}$. The center of mass of the piston is connected to a flywheel of moment of inertia $I$ at a distance $R$ from its center by a rigid and massless rod of length $\ell$, as shown. The system has only one degree of freedom but two natural coordinates, $\phi$ and $z$.

\[ \begin{array}{c}
\vec{g} \\
\ell \\
\phi \quad R \\
\quad I \\
z
\end{array} \]

(a) Express the Lagrangian in terms of $q_1 = z$, $q_2 = \phi$. [5 points]

(b) Write the constraint equation that connects the two coordinates. [5 points]

(c) From the above results, write down the two coupled equations of motion using the method of “undetermined multipliers”. Then eliminate the undetermined multiplier to obtain a single equation of motion (it can still involve both coordinates). [15 points]

(d) Find $p_\phi(z, \phi, \dot{\phi})$. [15 points]

*Hint:* A constraint equation of the form $C(q_i) = 0$ leads to Lagrange equation(s) of motion for the configuration variables $q_i$ that include additional terms $-\lambda \frac{\partial C}{\partial q_i}$, where $\lambda(q_i, t)$ is the undetermined multiplier.
NIU Physics PhD Candidacy Exam – Spring 2011 – Classical Mechanics

Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. A thin hollow cylinder of radius $R$ and mass $M$ slides across a rough horizontal surface with an initial linear velocity $V_0$. As it slides, it also has an initial angular velocity $\omega_0$ as shown in the figure. (Note that positive $\omega_0$ tends to produce rolling corresponding to motion in the direction opposite to $V_0$.) Let the coefficient of friction between the cylinder and the surface be $\mu$.

![Diagram of cylinder](attachment:diagram.png)

(a) How long will it take till the sliding stops? [16 points]
(b) What is the velocity of the center of mass of the cylinder at the time the sliding stops? [16 points]
(c) How do the results in (a) and (b) change for $\omega_0$ in the opposite direction from that shown in the figure? [8 points]

Problem 2. A block of mass $M$ is constrained to move without friction along a horizontal line (the $x$ axis in the figure). A simple pendulum of length $L$ and mass $m$ hangs from the center of the block. The pendulum moves in the $xy$ plane.

(a) [20 points] Find the Lagrangian and the equations of motion for the system.
(b) [20 points] Find the normal modes and normal frequencies of the system, assuming that the pendulum always makes a small angle with the vertical.
Problem 3. A uniform circular thin membrane with radius $R$ and mass/area $\mu$ is attached to a rigid support along its circumference, like a drumhead. Points on the membrane in the equilibrium position are labeled by polar coordinates $(r, \theta)$. The membrane has constant tension $C$, so that under a small displacement $f(r, \theta, t)$ from equilibrium, each area element $da$ contributes potential energy $\frac{1}{2}C(\nabla f)^2 da$.

(a) Find a wave equation satisfied by small time-dependent displacements $f(r, \theta, t)$ of the membrane from the equilibrium position. [16 points]

(b) Show that solutions can be found of the form $f = R(r)S(\theta)T(t)$, where:

\begin{align*}
\frac{d^2S}{d\theta^2} + k^2 S & = 0, \\
\frac{d^2T}{dt^2} + n^2 T & = 0,
\end{align*}

where $k$ and $n$ are constants, and $R(r)$ satisfies a second-order ordinary differential equation that you will determine. [16 points]

(c) Briefly describe how you would determine the vibrational frequencies of the membrane. [8 points]

Problem 4. A particle of mass $m$ moves subject to a central potential:

$$V(r) = -\frac{k}{r^n}$$

where $k$ and $n$ are positive constants.

(a) If the particle has angular momentum $L$, what is the radius $R$ for which the orbit is circular? [10 points]

(b) Suppose the motion is close to the circular orbit mentioned in part (a). Writing $r(t) = R + \delta r(t)$ and assuming that $\delta r(t)$ is small, find an equation for $\delta r(t)$. [15 points]

(c) Solve this equation for $\delta r$ and discuss the stability of circular orbits, for different values of $n$. [15 points]
Do ONLY THREE out of the four problems. Total points on each problem = 40.

Problem 1. Consider a solid cylinder of mass $m$ and radius $r$ sliding without rolling down the smooth inclined face of a wedge of mass $M$ and angle $\theta$, as shown. The wedge is free to move on a horizontal plane without friction.

(a) [16 points] How far has the wedge moved by the time the cylinder has descended from rest a vertical distance $h$?

(b) [16 points] Now suppose that the cylinder is free to roll down the wedge without slipping. How far does the wedge move in this case?

(c) [8 points] In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

Problem 2. Consider a binary system consisting of two small stars with comparable but unequal masses $m_1$ and $m_2$. The stars attract each other according to Newtonian gravity, and orbit each other at a fixed distance, with period $T$.

(a) [15 points] Find the distance between the stars.

(b) [25 points] Now suppose the motion of both stars is suddenly stopped at a given instant of time, and they are then released and allowed to fall into each other. Find the time that it takes for the stars to collide. (You may leave your answer in terms of a single definite integral over a real dimensionless variable.)
Problem 3. Consider a solid rectangular prism with mass $M$ uniformly distributed, and sides $a, a, b$, as shown.

<table>
<thead>
<tr>
<th>a</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td></td>
</tr>
</tbody>
</table>

Suppose that the object is suspended in a uniform gravitational field $g$ from one of its edges of length $a$, with that edge kept fixed and horizontal, and is free to swing as a pendulum.

(a) [20 points] Find a Lagrangian (with one degree of freedom) describing the motion.

(b) [10 points] Suppose that the pendulum starts from rest with its square side horizontal. Find the maximum speed reached by any part of the pendulum.

(c) [10 points] Find the angular frequency of small oscillations.
Problem 4. A mass-spring chain is formed by alternately connecting two kinds of masses, \(m\) and \(M\), by identical springs with spring constant \(k\), as shown below. Both ends of this mass-spring chain are fixed, and the springs are unstretched in equilibrium. There is no gravity. Assume that this system vibrates only longitudinally along the length of the chain. The masses are numbered from one end to the other, and the displacement of the \(n\)th mass from its equilibrium position is \(x_n\).

(a) [5 points] Find the equations of motion for the \(x_n\) describing the system, for \(n = 1, 2, \ldots, 2N + 1\), and write the boundary conditions for \(x_0\) and \(x_{2N+2}\).

(b) [20 points] Assume that \(x_n\) may be expressed as:

\[
x_{2j} = a_j e^{i\omega t} \quad (j = 0, 1, \ldots, N + 1)
\]

and

\[
x_{2j+1} = b_j e^{i\omega t} \quad (j = 0, 1, \ldots, N)
\]

where the \(a_j\) and \(b_j\) are time-independent. Express \(b_j\) in terms of \(a_j\) and \(a_{j+1}\), using the equations of motion obtained above. Also, what are \(a_0\) and \(a_{N+1}\)?

(c) [15 points] Using the results of part (b), find \(a_j\) in terms of \(a_{j-1}\) and \(a_{j+1}\). Now, assume that \(a_j\) can be expressed as \(a_j = A \sin(j\alpha + \phi)\), where \(A\), \(\alpha\) and \(\phi\) are constants independent of \(j\). Find the angular frequency \(\omega\). Also, determine the possible values of \(\alpha\) and \(\phi\).

Hint:

\[
\cos(x + y) - \cos(x - y) = -2 \sin x \sin y
\]
\[
\sin(x + y) + \sin(x - y) = 2 \sin x \cos y
\]
\[
\sin(x + y) - \sin(x - y) = 2 \cos x \sin y
\]
\[
\cos(x + y) + \cos(x - y) = 2 \cos x \cos y
\]
Problem 1. **Rocket Motion.** A rocket of initial total mass $m(0)$ at time $t = 0$ moves by ejecting gas at a variable rate $-\frac{dm}{dt} = \alpha(t)$. The gas is ejected at a constant velocity $u$ with respect to the rocket. The rocket moves vertically in a constant uniform gravitational field $g$ starting from rest on the ground at $t = 0$.

(a) Find the velocity of the rocket at time $t$, in terms of $u$, $g$, $m(0)$ and $\alpha(t)$. (Your answer may involve an integral.) [25 points]

(b) Find a condition on $\alpha(t)$ that is necessary and sufficient to guarantee that the rocket leaves the ground. [5 points]

(c) Now suppose that $\alpha$ is a constant. Find a set of specific algebraic equations whose solution determines the maximum height reached by the rocket. You do not need to solve the equations. [10 points]

Problem 2. **Spherical Pendulum.** A particle of mass $m$ is constrained to move on the surface of a sphere of radius $R$ in the presence of a uniform gravitational field $g$.

(a) Find the Lagrangian and the equations of motion in terms of spherical coordinates, with one of your coordinates being $\theta$, the polar angle measured from the bottom of the sphere. [15 points]

(b) Reduce the problem to an effective 1-dimensional problem in terms of $\theta$, for fixed angular momentum $L_z$, with energy $E = \frac{1}{2}mR^2\dot{\theta}^2 + V_{\text{eff}}$, and find $V_{\text{eff}}$. Sketch the corresponding effective potential $V_{\text{eff}}$, and use it to discuss the qualitative features of the motion. [10 points]

(c) As a special case, find the period for a horizontal circular orbit with the mass at a fixed polar angle $\theta_0$. [15 points]

Problem 3. **Scattering by a central potential.** Consider a particle of mass $m$ scattering from a central potential $V = -k/r^n$, where $k$ and $n$ are positive constants. The particle approaches from very far away with a non-zero impact parameter $b$ and initial velocity $v_0$.

(a) Show that for the particle to have a chance at hitting the origin ($r = 0$), it is necessary that $n$ is greater than or equal to a certain number that you will determine. [15 points]

(b) Now taking $n = 4$, show that a necessary and sufficient condition for the particle to hit the origin is that $b < b_{\text{crit}}$, where $b_{\text{crit}}$ is a quantity that you will determine in terms of $k$, $v_0$, and $m$. [20 points]

(c) Still taking $n = 4$, what is the cross section for particles to hit the origin? [5 points]
Problem 4. **Transverse motion of a string.**

A flexible, elastic string is initially stretched along the z axis with tension $T$ and mass per unit length $\mu$. The string has length $L$ and is fixed at both ends to solid walls.

(a) Derive the second-order differential equation for small transverse displacements $\psi(z, t)$ of the string in a fixed plane containing the z axis. Assume that each point on the string moves strictly at right angles to the z axis and that the tension at all points in the string remains close to its equilibrium value $T$. [10 points]

(b) What is the general solution for $\psi(z, t)$? Express it in the form of separation of variables with sinusoidal functions of $z$ and of $t$. [10 points]

(c) Now suppose that the middle point of the string is pulled a small amount $a$ perpendicular to the z-axis and held until released at $t = 0$. What are the initial conditions for $\psi$? [10 points]

(d) Apply the initial conditions of part (c) to the general solution of part (b) to obtain the complete solution $\psi(z, t)$. [10 points]

Possibly useful integrals:

\[
\int x \sin(x) \, dx = \sin(x) - x \cos(x)
\]
\[
\int x \cos(x) \, dx = \cos(x) + x \sin(x)
\]
\[
\int x \sin^2(x) \, dx = \frac{x^2}{4} - \frac{\cos(2x)}{8} - \frac{x \sin(2x)}{4}
\]
\[
\int x \cos^2(x) \, dx = \frac{x^2}{4} + \frac{\cos(2x)}{8} + \frac{x \sin(2x)}{4}
\]
\[
\int x \sin(x) \cos(x) \, dx = \frac{\sin(2x)}{8} - \frac{x \cos(2x)}{4}
\]
Problem 1. Three rough cylinders of equal radius are stacked as shown on a rough level surface. They have unequal masses $M_1, M_2, M_3$, but the coefficient of static friction is everywhere equal to $\mu$. How large must $\mu$ be so that the arrangement is stable? Assume that cylinders 2 and 3 do not quite touch.)

Problem 2. Suppose that the gravitational interaction between two objects with masses $M$ and $m$ is modified by replacing the potential term in the Lagrangian according to:

$$\frac{GMm}{r} \rightarrow \frac{GMm}{r} \left(1 + \frac{\dot{r}^2}{C^2}\right)$$

where $r = |\vec{r}|$ with $\vec{r}$ the relative position vector, a dot means a time derivative, and $C$ = a new constant. In the following, assume that $m \ll M$ and treat the heavier object as stationary.

(a) Find the equations of motion for orbital motion in appropriate variables. [12 points]

(b) Does the radial vector of the small orbiting object sweep out equal areas in equal times? (Explain briefly.) [4 points]

(c) Find the angular frequency of small oscillations about a circular orbit of radius $R$. Write your answer in terms of only $G, M, m, R,$ and $C$. (Do not assume that $C$ is small.) [12 points]

(d) Let $M_E$ and $R_E$ be the mass and the radius of the Earth (assumed to be a non-rotating perfect sphere). Compute the escape velocity for an object initially on the surface of the Earth, as a function of $G, M_E, R_E, C,$ and $\alpha = \text{the angle between the initial velocity at the Earth’s surface and the vertical}$. [12 points]
Problem 3. A very heavy flat-bed truck starts at rest on a level surface. It has a ball which is a solid sphere of radius $R$ and mass $M$ (with uniform mass density) on the bed as shown, with its center a distance $d$ from the back end. When the truck starts moving, the truck accelerates uniformly with acceleration $a_T$, and the ball rolls without slipping on the truck bed. (Note: moment of inertia of a uniform sphere about the center is $I = \frac{2}{5}MR^2$.)

(a) How long does it take for the ball to roll off of the truck? [30 points]
(b) What density distribution in the ball (with same total mass $M$ and radius $R$) would minimize the time to fall off? What is this minimum time? [10 points]

Problem 4. Two beads, with unequal masses $m_1$ and $m_2$, are constrained to slide frictionlessly on a stationary hoop of radius $R$. The beads are connected as shown by two identical springs with spring constant $k$ and unstretched length $d$. There is no gravity.

(a) Find the normal modes of the system and their angular frequencies. [25 points]
(b) Now suppose that at time $t = 0$ the beads are on opposite sides of the hoop, with bead $m_1$ having an instantaneous speed $v$ and bead $m_2$ at rest. Solve for the subsequent motion of the first bead. [15 points]
Problem 1 A uniform chain of length $L$ and total mass $M$ contains many links. It is held vertically by one end over a table with the other end just touching the table top.
(a) The chain is released and falls freely. What is the speed of the falling section of the chain at time $t$ after release? What is the force between the links? [10 points]
(b) Work out the increment of mass $dm$ that hits the table in an increment of time $dt$. Find the corresponding change in momentum and hence the instantaneous impulsive force on the table. [15 points]
(c) What is the total normal force acting on the table as a function of time? Show that the maximum value of the total force is $kMg$, where $k$ is a constant. Determine the value of $k$. [15 points]

Problem 2 Two identical thin uniform rods, each of length $d$ and mass $m$, are hinged together at the point $A$. The rod on the left has one end hinged at the fixed point $O$, while the end $B$ of the other rod slides freely along the horizontal $x$ axis. The system is in a uniform vertical gravitational field with acceleration $g$. All motion is frictionless.

(a) Find the total kinetic energy of the system. You should find the result:
\[ T = md^2 \dot{\phi}^2 (a + b \sin \phi + c \sin^2 \phi) \]
where $a$, $b$ and $c$ are constant numbers that you will determine. (Exactly one of $a$, $b$, and $c$ is 0.) [25 points]
(b) Suppose the system is at rest at time $t = 0$ with $\phi = \phi_0$. What is the velocity of the hinge $A$ when it hits the horizontal $x$ axis? [15 points]
Problem 3 A small spacecraft with mass $m$ and energy $E > 0$ approaches a star with mass $M$ and radius $R$ from far away.

(a) Find an expression for the effective total cross-section for the spacecraft to hit the star. [30 points]

(b) Sketch a graph of the cross-section as a function of $E$, and give simple physical explanations for the low and high energy limits. [10 points]

Problem 4 A wire in the shape of a parabola $z = \frac{r^2}{2a}$ rotates about the vertical $z$ axis with constant angular frequency $\Omega$. Here $r$ is the distance from the $z$ axis, and $a$ is a constant. A constant gravitational acceleration $g$ is directed in the negative $z$ direction. A small bead of mass $m$ slides on the wire without friction.

(a) Find the Lagrangian and obtain Lagrange’s equations of motion for the bead, using $r$ as the coordinate. [14 points]

(b) There exists a solution of Lagrange’s equations for which $\dot{r} = \text{constant}$. What is this constant? From where does the energy come to permit such unbounded motion? [8 points]

(c) Obtain the equations of motion for small deviations of the bead from rest at the bottom of the parabola. Give a condition for stable oscillation about this position. [8 points]

(d) Find the canonical momentum, the Hamiltonian, and Hamilton’s equations of motion for the bead. Is the Hamiltonian conserved? [10 points]
Problem 1  A spacecraft is placed in a nearly circular orbit about the Sun in the opposite direction but at nearly the same radius $R$ (and in the same plane) as an orbiting planet. Assume that $M_{\text{spacecraft}} \ll M_{\text{planet}} \ll M_{\text{Sun}}$, where the masses of the spacecraft, planet and sun are $M_{\text{spacecraft}}$, $M_{\text{planet}}$ and $M_{\text{Sun}}$. You should also assume that all motions are non-relativistic.

(a) What is the velocity of the spacecraft relative to the Sun when it is very far away from the planet? [7 points]

(b) What is the escape velocity of the spacecraft from the solar system? [7 points]

(c) Because the spacecraft and planet are nearly the same distance from the Sun and in the same plane, they will eventually have a close encounter. Suppose that the spacecraft encounters the planet (without colliding) and scatters at an angle $\theta$ in the rest frame of the planet. How large must $\theta$ be in order for the spacecraft to escape the solar system? [10 points]

(d) Use conservation of energy and angular momentum to find the total cross-section (with units of area) for the spacecraft to collide with the planet during a close encounter, if the planet’s radius is $a$. (Your answer should depend on other given quantities besides $a$.)
[16 points]
Problem 2  A particle is dropped from a tower attached to the Earth at latitude $\lambda$ from a height $h$ above the surface of the Earth. The Earth is rotating at constant angular velocity $\Omega$ counterclockwise looking down from the North Pole (the vertical line in the figure). The motion is to be described in a non-inertial set of coordinates fixed with respect to the Earth’s surface, in which the $z$ axis is an extension of the radius from the center, and the $x$ axis points along a meridian. (The $y$ axis is straight out of the page in the figure.) Assume the Earth is spherical and that $h$ is much smaller than the radius $R$.

(a) Find differential equations describing the motion of the particle, to lowest non-vanishing order in the angular velocity $\Omega$. [20 points]

(b) Compute the trajectory of the particle, again to lowest non-vanishing order in the angular velocity $\Omega$. What is the impact point where the particle hits the Earth? [20 points]
Problem 3  A spring pendulum consists of a mass \( m \) attached to one end of a massless spring with spring constant \( k \). The other end of the spring is tied to a fixed support. When no weight is on the spring, its length is \( l \). Assume that the motion of the system is confined to a vertical plane in the gravitational field of the Earth.

(a) Give the Lagrangian for this system. [14 points]

(b) Derive the equations of motion. [13 points]

(c) Find the general solution to the equations of motion in the approximation of small angular and radial displacements from equilibrium. [13 points]

Problem 4  A solid homogeneous cylinder of radius \( r \) and mass \( m \) rolls without slipping on the inside of a stationary larger cylinder of radius \( R \) in the gravitational field of the Earth as shown in the figure.

(a) How is the center-of-mass motion of the inner cylinder related with its rotational motion? [10 points]

(b) If the small cylinder starts at rest from an angle \( \theta_0 \) from the vertical, what is the total downward force it exerts on the outer cylinder as it passes through the lowest point? [10 points]

(c) Determine the equation of motion of the inside cylinder using Lagrangian techniques. [10 points]

(d) Find the period of small oscillations about the stable equilibrium position. [10 points]
Problem 1  A uniform cylinder of mass $M$ and radius $R$ is rolled up on an unstretchable massless string which is attached to the ceiling. The cylinder is released and unrolls the string as it falls (like a yo-yo).

Considering only vertical motion of the center of the cylinder, and assuming that the cylinder never reaches the end of the string:
(a) Specify the degrees of freedom and state any equations of constraint.  [5 points]
(b) Find the Lagrangian, and use it to solve for the motion as a function of time.  [25 points]
(c) Find the tension in the string as a function of time.  [10 points]

Problem 2  The extremely rare compound di-Cryptonite dioxide is composed of molecules that can be pictured as four particles (with two different masses $m$ and $M$) attached by springs (with two different spring constants $k$ and $K$) as shown in the diagram:

Find all of the normal mode frequencies of the molecule under the assumption that motion is only along the line joining the particles. For each normal mode, indicate by a set of four arrows the directions of motion of the particles at one instant in time.
Problem 3  Suppose the force of attraction between a point object of mass $M$ and a planet of mass $m$ (where $m \ll M$) is:

$$F = \frac{a}{r^2} + \frac{3b\ell^2}{r^4}$$

where $\ell$ is the angular momentum of the planet and $a, b$ are both positive constants. [Note: this does mimic the force of attraction between a planet and a black hole, in the non-relativistic limit, with $a = G M m$.]

(a) Under what conditions is a stable circular orbit possible? Give the radius in terms of the given parameters. [15 points]

(b) What is the smallest radius possible for any circular orbit as a function of $a$ and $b$, allowing for arbitrary $\ell$? (Hint: this occurs in the limit of very large $\ell$.) Is this circular orbit stable or unstable? [15 points]

(c) Find an expression for the angular frequency of small radial oscillations for the planet if it travels in a slightly non-circular orbit about the stable radius. [10 points]

Problem 4  A particle of unit mass is constrained to move under gravity (acting in the negative $z$ direction with constant acceleration $g$) on a smooth surface with height:

$$z = \frac{x^2}{2a} + \frac{y^2}{2b},$$

where $a$ and $b$ are positive constants. The surface is forced to rotate with constant angular velocity $\Omega$ about the $z$ axis.

(a) Find the equations describing small amplitude oscillations near the bottom of the surface, and show that they have the form:

$$\ddot{x} - c_1 \dot{y} + c_2 x = 0,$$
$$\ddot{y} + c_1 \dot{x} + c_3 y = 0,$$

where $c_1, c_2,$ and $c_3$ are distinct non-zero constants that you will find in terms of $\Omega, g, a,$ and $b$. [25 points]

(b) Find the frequencies of oscillatory solutions for these equations. [15 points]
A non-relativistic particle of mass $m$ moves in one dimension $x$ under the force

$$F(x) = ax^3 - bx,$$

where $a$ and $b$ are positive constants.

(a) Does a potential energy function exist for this force? Why? Is the total energy conserved? [6 points]

(b) Find an expression for the potential energy $V(x)$, and sketch its graph. Choose the arbitrary constant so that the potential energy vanishes at $x = 0$. [6 points]

(c) Find all of the turning points of the motion if the total energy is $E$. [8 points]

(d) For what values of the total energy $E$ is the motion bounded? [8 points]

(e) For positive values of $E$ less than that found in part (d), find the frequency for periodic oscillations. Do not assume that the oscillations are small. You may leave your answer in terms of a definite integral. [12 points]

Consider a pendulum hanging from the top of a car. The pendulum consists of a mass $m$ hanging from a massless rod of length $\ell$. Initially, the car is at rest and the pendulum is in its equilibrium position (aligned with the vertical). Now assume the car suddenly accelerates with constant horizontal acceleration $A_0$.

(a) Find the new equilibrium position's angle with respect to the vertical. [10 points]

(b) Find the differential equation governing the time dependence of the angle $\phi$ from the vertical. [10 points]

(c) What is the maximum angle through which the pendulum swings with respect to the vertical? (HINT: this is NOT the same as part (a).) [20 points]
Problem 3 The minimum distance of a comet from the sun is observed to be exactly half the radius of the earth’s orbit, and its speed at that point is twice the orbital speed of the earth. Ignore the effects of the earth and other planets on the comet, assume the mass of the comet is negligible compared to the mass of the sun $M$, and assume the earth’s orbit is circular with radius $R$ and in the same plane as the comet’s orbit.

(a) Find the velocity of the comet when it crosses the earth’s orbit and the angle at which the orbits cross. [16 points]
(b) Will the comet subsequently escape the solar system? Why? (No credit without valid reasoning.) [8 points]
(c) Calculate how long the comet remains inside the earth’s orbit. Give the answer in years. [16 points]

Problem 4 A thin, flat square of mass $M$ and side $a$ has a uniform mass density.
(a) Find the moments of inertia and write down the inertia tensor for a coordinate system with origin at the center of the square and $x, y$ axes parallel to the sides of the square and $z$ axis perpendicular to the square. [10 points]
(b) Suppose that the square is suspended frictionlessly by one fixed corner and allowed to oscillate as a pendulum in a constant gravitational field (with acceleration due to gravity $g$). The motion takes place within the same plane as the square. Find an equation of motion for the system, and use it to find the angular frequency for small oscillations. [15 points]

(c) Repeat part (b), but now assuming the oscillating motion is along a direction normal to the plane of the square. [15 points]
Problem 1. A ball of mass $m$ and radius $R$ (moment of inertia $I = \frac{2}{5}mR^2$ about its center) is released from rest at an angle $\theta_0$ in a stationary spherical bowl of radius of curvature $d + R$. The ball rolls on the bowl without slipping (in the plane of the diagram).

(a) [25 points] Derive an equation of motion for the angle $\theta$ through the center of the ball, as defined in the figure. Solve this equation of motion for the case of $\theta_0 \ll 1$.

(b) [15 points] What is the velocity of the center of the ball when $\theta = 0$? (Do not assume $\theta_0 \ll 1$ here.)
Problem 2  An isotropic oscillator potential $V(r) = \frac{1}{2}kr^2$ leads, like the gravitational $-1/r$ potential, to closed planar orbits. Suppose, however, that the “spring” has a non-zero relaxed length $a$, so that:

$$V(r) = \frac{1}{2}k(r-a)^2$$

and consider the (planar) motions of a particle of mass $m$ and initial angular momentum $L$ in this potential.

(a) [20 points] Find a condition relating the radius $r_0$ of circular orbits and the given constants. (It is not necessary to solve for $r_0$.)

(b) [20 points] For nearly circular orbits, find the angular change between successive radial maxima. For $a/r_0 \ll 1$, express your answer to first order in $a/r_0$ (eliminate $L$ from the answer). Check that the orbits are indeed closed for $a/r_0 \to 0$

Problem 3 [40 points] Consider a system of masses $m$ and $2m$ and springs with spring constants $k$, $k/2$, and $2k$, attached between two stationary walls as shown. For motions along the axis of the springs, describe the normal modes of vibration and find their angular frequencies.
Problem 4  A thin, flexible, unstretchable loop of string has mass per length $\mu$. It is set spinning with angular frequency $\Omega$ in a gravitationless vacuum in such a way that it forms a stable circle of radius $R$.

(a) [20 points] Find the tension in the string.

(b) [20 points] Now suppose the string is given a small tap so that infinitesimal vibrations travel along the loop in the plane of the string in both directions. Compute the speeds of those waves relative to a non-rotating coordinate system.
Problem 1  A small planet of mass $m$ revolves around a star of mass $M$ in a nearly circular orbit with slightly varying separation, $r$. You may neglect $m$ in comparison to $M$. Centered on the star is a spherical dust cloud of uniform density $\rho$, with radius large enough to contain the orbit of the planet. (The planet does not lose energy due to collisions with the dust cloud.) Let $G = \text{the gravitational constant}$. 

(a) [14 points] If the orbit were exactly circular with $r = R$, what would be the angular frequency $\omega \equiv d\theta/dt$ and the angular momentum $L$? (Give your answers in terms of $m$, $M$, $R$, $\rho$, and $G$.)

(b) [14 points] The equation of motion for $r$ is the same as that of an equivalent one-dimensional problem. For that problem, find the effective potential. (Give your answer in terms of $m$, $M$, $\rho$, $G$, the angular momentum $L$, and $r$.)

(c) [12 points] Consider the case of small $\rho$, and small deviations from the circular orbit, $r(t) = R + \epsilon(t)$. Find the angular velocity of the precession of the perihelion, $\omega_p = \omega_r - \omega$, and show that it can be written as

$$a(GR^3/M)^c \rho,$$

where $a$ and $c$ are non-zero quantities that you will determine.

Problem 2  [40 points] A particle of mass $m$ slides from the very top of a sphere of radius $R$, with initial horizontal velocity $v$, under the influence of a uniform gravitational field. The coefficient of friction between the particle and the sphere is $\mu$. (This means that the force of friction opposing the motion is $\mu N$, where $N$ is the force normal to the surface of the sphere.) The constant acceleration of gravity downward is $g$. What is the minimum value of $v$ for which the particle will fall off of the sphere?
Problem 3  A projectile is launched from the ground at an angle of 45 degrees and with an initial kinetic energy $E_0$. At the top of its trajectory, the projectile explodes into two fragments with masses $m_1$ and $m_2$. The explosion imparts an additional mechanical energy $E_0$ to the system. The fragment of mass $m_1$ is then observed to travel straight down. Assume the motion is in the $xy$ plane, with $x$ horizontal and $y$ vertical, and the acceleration of gravity downward is $g$.

(a) [20 points] Find both of the components $v_{2x}$ and $v_{2y}$ of the velocity vector of the mass $m_2$, and the magnitude $v_1$ of the velocity of the mass $m_1$, immediately after the explosion. What is the maximum possible value of $m_1/m_2$?

(b) [20 points] Find an expression for the horizontal range of $m_2$, measured from the initial launch position, in terms of the given quantities.

Problem 4  Two rigid thin rods of uniform mass density each have mass $M$ and length $L$. They are pivoted freely at their top ends, and joined by a spring at their bottom ends, as shown in the figure below. The rods are constrained to move in the plane of the page. The spring has spring constant $k$ and length $b$ when unstretched, and the distance between the pivot points is also $b$. The acceleration due to gravity is $g$.

(a) [16 points] Find the Lagrangian for this system.

(b) [12 points] Find the equations of motion for small oscillations of this system.

(c) [12 points] Sketch the motions corresponding to the normal modes. Find the corresponding normal frequencies for small oscillations.

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![Diagram of the system with rods and a spring]
Do any THREE out of four problems

Problem 1  A spherical star of a definite radius is constructed from an incompressible fluid of constant mass density $\rho$. The star is held together by its own gravitational attraction. The total mass of the star is $M$. Find the pressure $P(r)$ within the star as a function of the distance from the center, assuming that the star is not rotating.

Problem 2  A particle of mass $m$ is confined to move on the frictionless surface of a right circular cone whose axis is vertical, with a half opening angle $\alpha$. The vertex of the cone is at the origin and the axis of symmetry is the $z$ axis. For a given non-zero angular momentum $L$ about the $z$-axis, find:
(a) the height $z_0$ at which one can have a uniform circular motion in a horizontal plane.
(b) the frequency of small oscillations about the solution found in part (a).
Give your answer in terms of only $m$, $\alpha$, $L$, and the acceleration due to gravity $g$.

Problem 3:  A thin uniform disk of radius $a$ and mass $m$ is rotating with a constant angular velocity $\omega$ about a fixed axis passing through its center but inclined at an angle $\alpha$ with respect to the axis of symmetry of the disk.
(a) Find the magnitude of the angular momentum vector about the center of the disk. 
[Hint: the moments of inertia of the disk about its principal axes satisfy $I_1 = I_2 = \frac{1}{2}I_3$.]
(b) Find the torque that is exerted about the center of the disk to make this happen.
Give your answers in terms of $a, m, \omega, \alpha$ only.

Problem 4:  A CO$_2$ gas molecule is linear, as shown in the figure below, with its long molecular axis in parallel with the $x$-axis. The equilibrium distance between the carbon atom and each of the two oxygen atoms is $a$. The spring constant of each CO bond is $k$. The mass of an oxygen atom is $M$, and the mass of a carbon atom is $m$. Using the Lagrangian method, find all of the normal modes of the molecule that have motion only along the $x$-axis, and describe their motions. Ignore the sizes of the atoms.
1. Consider a projectile fired from the origin with a velocity \( \vec{v}_0 = i v_{0x} + k v_{0z} \) where \( z \) is the vertical coordinate. There is a wind with velocity \( \vec{v}_w = j v_w \) and air resistance is proportional to velocity \( \vec{F}_{d} = -b \vec{v} \).
(a) Find the position as a function of time for all three coordinates.
(b) Find the displacement in the \( x \)-position due to air resistance at \( z=0 \) keeping only the first order terms in \( b \).
(c) Find the displacement in the \( y \)-position due to the wind at \( z=0 \) keeping only the first order terms in \( b \).

2. Suppose at a latitude \( \lambda = 20^\circ \text{N} \) the atmospheric pressure is \( P = 10^5 \text{ N/m}^2 \) and the air density is \( \rho = 1.3 \text{ kg/m}^3 \).
(a) Determine an expression for the velocity \( v \) as a function of the pressure gradient and the radius \( r \).
(b) Use the result in part (a) to find the wind speed for a low pressure region with a pressure gradient of 3 millibar/m at 100 km from the center of low pressure.

3. Consider the motion of a particle of mass \( m \) moving on the outer surface of a hoop of radius \( R \). The particle is subject to the force of gravity on the Earth’s surface \( mg \). Use polar coordinates \((r, \theta)\) as generalized coordinates to describe the motion of the particle.
(a) Find the Lagrangian for the particle in terms of the generalized coordinates.
(b) Use Lagrange’s equations to find an expression for the force of the constraint.
(c) Determine the angle at which the particle leaves the surface of the hoop.

4. A uniform string of length \( L \) and linear mass density \( \mu \) under tension \( T \) is displaced initially at rest as shown below.

(a) Write expressions for the initial conditions of the string.
(b) Find a general solution of the vibrating string as a Fourier series.
(c) Use the initial conditions to find the coefficients of the Fourier series.
1. A flexible but unstretchable rope has a length $L$ and has a mass-per-unit-length $\lambda$. It is held at rest on a smooth frictionless horizontal surface with a length $x_0$ hanging vertically over the edge. The rope is released at time $t_0$ and slides off the surface.

(a) Find the speed of the rope at any time $t$ after it is released, but before it leaves the surface.

(b) Find the time at which the rope leaves the surface.

2. A comet is observed at a distance of $10^8$ km from the center of the sun. At that point the comet is traveling with a total velocity of 56.6 km/s with equal components $v_x$ and $v_y$ where $x$ is the direction toward the sun.

(a) Find the angular momentum per unit mass of the comet, and the total energy per unit mass of the comet ($GM_{\text{sun}} = 1.33 \times 10^{11}$ km$^3$/s$^2$).

(b) Find the eccentricity of the orbit of the comet, and identify if the orbit is open or closed.

(c) Find the distance of closest approach between the comet and the center of the sun.

3. Consider the motion of a particle of mass $m$ moving in a plane. The particle is subject to a force $\mathbf{F} = iF_x + jF_y + kF_z$. Use cylindrical coordinates ($\rho, \theta, z$) as generalized coordinates to describe the motion of the particle.

(a) Find the changes in the Cartesian coordinates $\delta x, \delta y, \delta z$ in terms of the generalized coordinates.

(b) Find the generalized forces $Q_{\rho}, Q_{\theta},$ and $Q_z$ associated with the generalized coordinates.

4. A light rod of length $r$ is fixed at the origin, and a mass $M$ is attached to the other end, as shown at right. The rod is constrained to move in the $XY$-plane. A pendulum of length $l$ and mass $m$ attached at $A$ can oscillate in the $YZ$-plane. Use $\theta$ for the angle of the rod in the $XY$-plane, and $\phi$ for the angle of the pendulum in the $YZ$-plane.

(a) Find Lagrange’s equations for the system of the rod and pendulum in terms of $\theta$ and $\phi$.

(b) Find the normal frequencies and normal modes of vibration for small oscillations.
1. A mass-spring ring system as shown at right consists of \( N \) identical masses, \( m \), and \( N \) springs with spring constant \( k \).

(a) Show the equation of motion of this system when the masses move along the circle of this mass-spring system.

(b) Find the amplitude mode angular frequency of the mode in part (a). \( \omega = \sqrt{\frac{k}{m}} \)

2. Consider two pendula of equal length \( b \) and equal mass \( m_1 = m_2 = m \) connected by a spring of force constant \( k \) and both constrained to move in the same plane. The spring is unstretched when the system is in its static equilibrium configuration, and the pivot points are separated by a distance \( L \).

(a) Write down the Lagrangian for the system. Do not assume small-amplitude motion. How would you determine the equations of motion.

(b) Now, in the limit of small-amplitude motion, the equations of motion simplify to:

\[
\frac{d^2}{dt^2} (\theta_1) + \left( \frac{g}{b} + \frac{k}{m} \right) \theta_1 - \frac{k}{m} \theta_2 = 0
\]

\[
\frac{d^2}{dt^2} (\theta_2) + \left( \frac{g}{b} + \frac{k}{m} \right) \theta_2 - \frac{k}{m} \theta_1 = 0
\]

Using normal coordinates \( \theta_1 + \theta_2 \) and \( \theta_1 - \theta_2 \), derive the corresponding eigenfrequencies \( \omega_1 \) and \( \omega_2 \).

3. A fluid of density \( \rho \) and viscosity \( \eta \) flows at a constant rate between two plates separated by a distance \( L \). The total fluid flow per unit length between the walls perpendicular to the direction of current is \( j \). Assume that the pressure varies only in the direction of flow.

(a) Find the component of velocity parallel to the walls as a function of the distance from the midpoint between the plates and the pressure gradient parallel to the walls.

(b) Find a relationship between the pressure gradient and the fluid flow per unit length.

4. A cable is hung at equilibrium with the end points at \((x_A, y_A)\) and \((x_B, y_B)\). The cable is supporting a load that is uniformly distributed in the horizontal direction where \( w \) is the weight per unit length. The tension of the cable at the lowest point is \( T_0 \). Find an expression for the span \( L = x_B - x_A \) as a function of \( w, T_0, y_A \) and \( y_B \).
1. Consider an infinitely long string, as shown below. For \( x < 0 \) and \( x > L \), the linear mass density of the string is \( \mu_1 \), and for \( 0 < x < L \), the linear mass density is \( \mu_2 (\mu_2 > \mu_1) \). A wave of amplitude \( A_0 \) and frequency \( \omega \) is incident from the left side. Find the reflected and transmitted intensities at A and B.

\[
\begin{array}{c|c|c|c}
 & A & B & \\
\hline
\mu_1 & & & \\
x = 0 & \mu_2 & & \\
x = L & & \mu_1 & \\
\end{array}
\]

2. A door on frictionless hinges is hung at a slight angle \( \theta \) with respect to the vertical. Calculate the moment of inertia for the door and use it to write the Lagrangian for the door. Find Lagrange's equation of motion for the door and use it to determine the period of small oscillations. If it takes 1 s for the door to close when slightly ajar, what is the angle of the door with respect to the vertical?

3. Consider two wells filled with water dug entirely to the center of the earth. One well is directly under the moon at the equator and the other well is at the equator but perpendicular to the first well. Assume that the water in the well is incompressible and that the pressures at the top of the two wells are equal and also the pressures at the bottom of both wells are equal. Write an expression for the pressure at the bottom of the well as a function of the weight of the water. Use this expression to determine the difference in the height of the water in the two wells. Newton used this method to estimate the tides.

4. A tippie top is a nearly spherical top of mass \( M \), that will flip when spinning such that the heavy end is aligned upwards. Assume that the center of mass is only slightly below the center of curvature of the sphere, and that all three moments of inertia are equal to the moment of inertia for a sphere.

(a) Consider the body-centered motion of the top with the z-axis along the spindle and draw the forces, torques, angular momentum and angular velocity on the top.

(b) Write an expression for the force of friction on the table, and the angular equation of motion.

(c) If \( \theta \) is the angle between the angular momentum and the z-axis of the top, find an expression for \( d\theta/dt \). If the top is 1 cm in radius, spun at 300 rad/s and the coefficient of friction is 0.1, find the time it takes to flip.
1. A mass $M_2$ hangs at one end of a string which passes over a fixed frictionless, non-rotating pulley. At the other end of the string there is a frictionless non-rotating pulley of mass $M_1$ over which there is a string carrying masses $M_3$ and $M_4$.

(a) Set up the Lagrangian of the system.
(b) Use Lagrange’s equation to find the acceleration of mass $M_2$.

2. A rocket of initial mass $m_0$ moves by ejecting gas at a constant rate $\alpha$. The gas is ejected at a constant velocity $u$ with respect to the rocket. The rocket is moving vertically upward in a constant gravitational field of magnitude $g$.

(a) Write Newton’s law of motion for the rocket.
(b) Find the velocity of the rocket as a function of time in terms of the constants given above.
3. A disk of mass $M$ and radius $R$ is attached to the end of a rod of mass $M$ and length $L$ as shown below. The rod is free to swing without friction from the opposite end that holds the disk, and the disk is free to rotate without friction at its center.

![Diagram](image)

(a) Find the moments of inertia of the rod, disk, and combined system at the pivot of the rod.
(b) Find the Lagrangian of the system in terms of the two angles $\theta$ and $\phi$.
(c) Find Lagrange's equations of motion, and compare the motion to that of a simple pendulum.

4. Consider an isotropic harmonic oscillator whose potential is given by $V(r) = \frac{1}{2} kr^2$.
(a) Derive the effective potential $V_{\text{eff}}(L, r)$ for a particle of mass $m$, and make a plot of $V_{\text{eff}}(r)$ versus $r$.
(b) Find the values of the energy $E$ and angular momentum $L$ for a circular orbit and identify that point on the graph from part (a).
(c) Find the frequency of revolution for the circular orbit of part (b) and the frequency of small radial oscillations.
Solve 3 out of 4 problems.

I. A marble of mass \( m \) is sliding (not rolling) down the side of a hemispherical dish with a radius \( b \). Find Hamilton’s equation of motion for the marble.

II. The differential equation of motion describing the displacement from equilibrium for damped harmonic motion is:
\[
m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0
\]

a. State conditions and describe the motion for overdamping, critical damping, and underdamping.

b. Show that the ratio of two successive maxima in the displacement \( x \) is constant.

III. Consider a point particle which moves in a central potential subject to a radial force
\[
F_r = -\frac{a}{r^3} - \frac{b}{r},
\]
with \( a, b \), both positive. At the origin there is a particle eater which swallows any particle that reaches \( r = 0 \). The point particles starts out at \( r = R \) with angular momentum \( L \) and initial radial velocity \( \frac{dr}{dt} = v_r \). Enumerate the conditions on \( R, v_r, \) and \( L \) that will result in the particle being eaten.

IV. A homogeneous cube of mass \( M \) and edge length \( L \) is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a tiny displacement and allowed to fall.

a. Find the angular velocity of the cube when one face strikes the plane, assuming the edge slides without friction.

b. Same question as (a), but now assuming the edge cannot slide because of friction.

c. For case (a), find the force exerted by the surface on the cube just before the face strikes the plane. (The moment of inertia of a cube about an axis through its center and parallel to an edge is \( I = ML^2/6 \).)