ABSTRACT

The accelerator and beam physics community continues to unravel the underlying mechanisms that lead to particle beam degradation and their corresponding time scales. This ongoing research is driven by a necessity to design and create high-current, high-intensity accelerators and transport systems that will become the basis for future research. Thus, research groups are currently devising theoretical models and experiments that peer into such fundamental complexities. The particle-core model was an early, successful first step towards defining the structure behind mechanisms that deteriorate beam quality.

The results reported herein draw from numerical studies that elaborate on a family of particle-core models that emulate an intense charged-particle beam bunch. This study emphasizes the role of time dependence and chaotic mixing using several waterbag particle-core models to analyze mechanisms of collective beam instabilities such as halo formation and emittance growth that can lead to beam loss.
NORTHERN ILLINOIS UNIVERSITY

CHAOS IN TIME-DEPENDENT SPACE-CHARGE POTENTIALS

A THESIS SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE
MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

BY
GREGORY T. BETZEL

©2005 Gregory T. Betzel

DEKALB, ILLINOIS
DECEMBER 2005
ACKNOWLEDGEMENTS

The author wishes to express the utmost appreciation and gratitude to Dr. Courtlandt Bohn and Dr. Ioannis Sideris for their guidance and encouragement. In addition, a special thanks to everyone at NICADD and the NIU Physics Dept. who helped support my efforts.
DEDICATION

To Amycla
# TABLE OF CONTENTS

LIST OF FIGURES............................................................................................................. vii

Chapter

1. INTRODUCTION........................................................................................................ 1

2. THEORETICAL TOOLS......................................................................................... 3

   2.1 Statistical Mechanics of a Beam Bunch......................................................... 3

   2.2 Chaotic Motion............................................................................................... 5

   2.3 Noise.............................................................................................................. 8

   2.4 Conventional Measures: Poincaré Mapping, Emittance, and Energy Spread......................................................... 9

   2.5 The Particle-Core Model............................................................................... 11

3. NUMERICAL METHODS...................................................................................... 16

   3.1 Summary.......................................................................................................... 16

   3.2 Integration Methods....................................................................................... 17

       3.2.1 The Runge-Kutta Method................................................................. 17

       3.2.2 Lyapunov Exponents.............................................................. 18

       3.2.3 Generating Colored Noise.......................................................... 19

   3.3 Simulation Parameters.................................................................................. 19

4. RESULTS AND ANALYSIS................................................................................. 22

   4.1 Overview......................................................................................................... 22

   4.2 Halo Formation.............................................................................................. 23
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Particle orbit trajectories and power spectra for one regular and one chaotic orbit</td>
<td>6</td>
</tr>
<tr>
<td>2.2</td>
<td>Waterbag density profiles</td>
<td>14</td>
</tr>
<tr>
<td>3.1</td>
<td>Renormalization of Lyapunov exponent</td>
<td>19</td>
</tr>
<tr>
<td>4.1</td>
<td>A particle orbit trajectory and parametric resonance</td>
<td>24</td>
</tr>
<tr>
<td>4.2</td>
<td>PSS of a wildly chaotic orbit in the presence of colored noise</td>
<td>25</td>
</tr>
<tr>
<td>4.3</td>
<td>Lyapunov exponents vs. initial condition $r$, with tune $\eta = 0.15$ and mismatch $M = 0.50, 10,000$ orbits</td>
<td>26</td>
</tr>
<tr>
<td>4.4</td>
<td>Stroboscopic PSS snapshots of mixing in a breathing spherical waterbag potential with considerable space charge, $\eta = 0.15$</td>
<td>27</td>
</tr>
<tr>
<td>4.5</td>
<td>Percentage orbits in halo vs. $t_D$</td>
<td>29</td>
</tr>
<tr>
<td>4.6</td>
<td>Percentage orbits in halo vs. $t_D$ for two comparative waterbags</td>
<td>30</td>
</tr>
<tr>
<td>4.7</td>
<td>Natural logarithm of the rms emittance $\ln(\varepsilon_{rms})$ and energy spread $\ln(</td>
<td>\Delta E</td>
</tr>
<tr>
<td>4.8</td>
<td>Stroboscopic plots exhibiting early exponential growth</td>
<td>34</td>
</tr>
<tr>
<td>4.9</td>
<td>Poincaré sections for two comparative spherical waterbags</td>
<td>36</td>
</tr>
<tr>
<td>4.10</td>
<td>Lyapunov exponents of 10,000 orbits vs. initial condition $r$, with tune $\eta = 0.15$ and mismatch $M = 0.50, 0.75$, for various noise strengths</td>
<td>37</td>
</tr>
<tr>
<td>4.11</td>
<td>Average Lyapunov exponents and percentage of chaotic orbits for samples of 10,000 initial conditions using tunes $\eta = 0.15, 0.25$ for various mismatch parameters and various noise strengths</td>
<td>38</td>
</tr>
<tr>
<td>4.12</td>
<td>A particle trajectory and its corresponding PSS for a “sticky” chaotic orbit</td>
<td>40</td>
</tr>
</tbody>
</table>
Figure 4.13  The complete PSS of the very “sticky” chaotic trajectory in Figure 4.11.......................................................... 41

4.14  $x$ vs. $t_D$ for three distinct orbits, before and after the presence of noise.. 42

4.15  Maximum amplitude vs. $M$, various noise strengths with a collective mode......................................................... 45

4.16  PSS for $\eta = 0.25$, $M = 0.75$, $\varepsilon = 0.05$, with increasing noise strengths $\langle |\delta \omega| \rangle = 0, 0.002, 0.01$, with a nonlinear focusing force...................... 47
CHAPTER I

INTRODUCTION

Future high-power proton linear accelerators (linacs) for the transmutation of waste, accelerator-driven neutron spallation sources, and accelerators for nuclear production must be designed to effectively control beam instabilities. Among numerous variables that may interfere with beam stabilities, halo formation appears to be the Achilles tendon that leads to unacceptable machine radioactivation. The decisive limit for such uncontrolled beam losses that would require routine, hands-on maintenance is ~1 W/m. For instance, losses for a 100 mA, 1 GeV light-ion beam would need to be kept to 1 in $10^8$ particles per meter.\(^1\)

Early investigations\(^2\) considered a two-parameter family of spatially uniform density distributions, which were made to pulsate from an initial beam size mismatch,\(^3\) to explore defined space-charge effects and the mechanisms that deteriorate beam quality, especially halo formation in high-current linacs. These particle-core models were designed to allow test particles to evolve as they passed through a central charge

---

\(^3\) “Mismatched” beams are not synchronized with the focusing system, producing time-varying space-charge forces that can transfer energy to some of the particles. See, e.g., Wangler, T.P., *RF Linear Accelerators* (John Wiley & Sons, New York, 1998), 217, 286.
distribution, or core, each particle “feeling” the time-dependent force of the core with a consequent change in energy. Test particles were tracked throughout the evolution of the bunch, enabling an observer to characterize the distribution without the test particles contributing to the total potential.

Subsequent analytical and numerical studies concentrated on various aspects of the particle-core model. An important conclusion was that single particles interacting with the beam core via parametric resonance are driven out of the core to large amplitudes, creating a halo and increasing the risk of beam loss and radioactivation. One critical prediction postulated a hard upper bound to the amplitude of the halo particles. However, recent studies have shown that introducing noise into the system creates a continual source of halo production, ejecting orbits to ever-growing amplitudes. This study also introduces noise to illustrate the impact on halo formation and the fragility of tori that would otherwise confine the beam.

The research herein makes use of a particle-core model to utilize the evolutionary dynamics of a space-charge-dominated beam bunch. Specifically, a spherically symmetric, homologously breathing, waterbag model is considered to investigate the underlying mechanisms of halo formation and other instabilities inherent in such a configuration. It will be revealed that time dependence alone triggers complicated evolutionary dynamics.

---

4 The initial study used an azimuthally symmetric Kapchinkij-Vladimirskij (KV) core distribution.
CHAPTER II
THEORETICAL TOOLS

2.1 Statistical Mechanics of a Beam Bunch

Through detailed analyses of the evolution in space-charge potentials, the microscopic properties of particle distributions are fairly well understood. However, the assumptions and approximations leading to the macroscopic properties of charged-particle beams are often applied tacitly. Here, this process is reviewed for completeness, as it will be the foundation for the model to be analyzed.

Microscopic dynamics require the consideration of every individual particle, so that a beam bunch may be described as an $N$ particle distribution interacting in $6N$-dimensional phase space. Here, even in the presence of a time-dependent potential, each point in phase space moves according to the equations of motion, consistent with Liouville’s theorem:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{x} \cdot \frac{\partial f}{\partial x} + \dot{p} \cdot \frac{\partial f}{\partial p} = 0,$$

where $f(x,t)$ is the distribution of particles in the $6N$-dimensional phase space $x$. The idea, though, is to follow the evolution of the distribution function rather than the

---

motion of the single particles. A careful development of statistical dynamics is needed in order to ensure that this distribution sufficiently models macroscopic properties.

The distribution of mutually interacting identical particles may be defined using the BBGKY hierarchy of reduced distribution functions, but the formalism is useless unless some physically sensible truncation scheme is applied. By recasting the distributions in terms of correlation functions, the hierarchy can be truncated so that the result yields the Vlasov-Poisson equation, or the collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \frac{p}{\gamma m} \cdot \frac{\partial f}{\partial r} + q(E + \frac{p \times B}{\gamma m}) \cdot \frac{\partial f}{\partial p} = 0$$

Note that $f$ is now in six-dimensional phase space.

The motion of these orbits in a real accelerator not only depends on the externally applied fields but also on Coulomb interactions and induced current on the accelerating structure. For this particular model, the effect of single Coulomb interactions is not important. A rigorous treatment of Coulomb scattering of a self-consistent 6-D distribution for a spherical bunch helped argue that single Coulomb collisions do not significantly contribute to emittance growth or halo development in linear accelerators. This should not, however, be confused with the space-charge field, resulting from combining multiparticle fields.

Presuming that particle correlations are unimportant, the Vlasov-Poisson equations are used to calculate the distribution function of the six-dimensional phase space of a particle. The charge density and space-charge potential are coarse grained; that is, granularity of the system has been removed. This distribution is the basis of a

---

10 Ibid., 60.
model that will contribute to a picture of the macroscopic properties of a charged-particle beam. The dynamics of these macroscopic properties may be readily understood in terms of mixing, and this mixing depends on the nature of the orbits, that is, whether they are regular or chaotic.

2.2 Chaotic Motion

This study uses the particle-core model to perform mixing experiments by carefully tracking and observing the behavior of test particles. The test particle trajectories we especially seek to utilize behave chaotically. Thus, it is important to understand chaos and how chaotic behavior may be measured.

Chaotic orbits arise only when the stationary waterbag is made to oscillate, or breathe. Once the oscillations due to time dependence are removed, the equation of particle motion is integrable up to quadrature because this is then a one-dimensional system. All orbits would be regular in the absence of time dependence. Regular orbits are well-behaved periodic orbits that are confined to restricted motion. Particles that are sensitive to initial conditions and follow much more complex orbits are labeled as chaotic. Figure 2.1 illustrates the clear distinction between a regular and a chaotic orbit. The power spectra clearly show that the regular orbit exhibits (three) distinct frequencies in its spectrum, whereas the power spectrum of the chaotic orbit is complicated and exhibits continua. The irregularity of chaotic behavior must not be confused with random motion, for the solutions to the equations of chaotic motion are, in fact, deterministic. However, they are not predictable. Technically, chaotic systems
are predictable, but only on a short time scale. For long time scales, chaotic systems become unpredictable, at a point - called the prediction horizon - that depends on the uncertainty of the initial conditions. Most importantly, because the initial conditions are sensitive to change, a minute deviation of position and velocity results in considerable divergence from nearby orbits. The largest Lyapunov exponent, $\chi$, quantifies this exponential divergence, distinguishing among orbits, either stable or unstable.

![Figure 2.1. Particle orbit trajectories (left) and power spectra (right) for one regular (top) and one chaotic (bottom) orbit. The Lyapunov exponent $\chi$ is used to compare chaoticity.](image)

$^{12}$ Lyapunov may also be spelled Liapunov or Ljapunov.
Lyapunov exponents\textsuperscript{13} can be used to measure chaoticity and measure how particle orbits explore different regions in phase space. This mean exponential rate of divergence $\chi$ (or convergence) of two localized trajectories, $x_0$ and $(x_0 + \Delta x)$, is defined as the limit

$$
\chi = \lim_{t \to \infty} \lim_{|\Delta x| \to 0} \frac{1}{t} \ln \frac{|\Delta x(X_0, t)|}{|\Delta x_0|},
$$

where $X_0$ is the initial position of the reference orbit. Notice that the separation of the two orbits $\Delta x$ is a function of both time and space. Not only do the orbital positions change temporally, but we also assume that the sensitive spatial dependence on the initial conditions may only arise in a fraction of the system in question. In addition, the separation must also be localized; otherwise, the deviation that follows is meaningless. Thus, the spatial limit is also considered.

The resulting average logarithmic rate of separation falls into one of three categories: a value of $\chi > 0$ indicates how much two localized orbits diverged, the average growth of that deviation arising from some perturbation, and, most importantly, associated with chaotic motion; an $\chi = 0$ exhibits stability and is therefore associated with regular orbits; and an $\chi < 0$ is associated with particles approaching some stable region. A negative $\chi$ can illustrate a dissipative system such as a damped harmonic oscillator.

2.3 Noise

A significant modification to the parameters that reshapes evolutionary dynamics is the addition of noise. Unavoidable machine errors as well as ordinary irregularities in the beam line impart a time-dependent noise on the orbits, which justifies the exploration of the effects of noise in this model. Previous studies have shown that there is in principle no hard upper bound to the halo amplitude given the presence of colored noise\(^{14}\) or statistical gradient errors.\(^{15}\) This study includes colored noise as well in order to model frequency fluctuations in the equations of motion of a particle orbit. A recent investigation\(^{16}\) has shown that a 1% rms focusing error translates into a few percent rms fluctuation in the envelope oscillation frequency.

Gaussian “white” noise refers to a normally distributed stochastic process.\(^{17}\) This type of noise may be used to model discreteness effects associated with close encounters within a beam. Here, Gaussian “colored” noise\(^{18}\) is used, as it readily mimics smoother, time-correlated fluctuations associated with a real beam line, such as those generated by machine imperfections.

Following an earlier investigation,\(^{19}\) Gaussian colored noise is added to the core oscillation frequency \(\omega\) such that

\[
\omega \to \omega(t) = \omega + \delta\omega(t),
\]

\(^{16}\)Ibid.
\(^{18}\)Ibid., 159.
\(^{19}\)Bohn & Sideris, op. cit., 264801.
where $\delta \omega(t)$ samples an Ornstein-Ulhenbeck process. Note that the first two moments of $\delta \omega(t)$ determine the noise characteristics:

$$\langle \delta \omega(t) \rangle = 0, \langle \delta \omega(t_1) \delta \omega(t_2) \rangle \propto e^{-\frac{|t_1 - t_2|}{t_c}},$$

in which $t_c$ is the autocorrelation time, the time scale over which the signal changes appreciably. Note that in the limit where $t_c$ goes to zero, this noise degenerates into white. User-defined inputs of Gaussian white noise and autocorrelation time are used to calculate colored noise in this study.

### 2.4 Conventional Measures: Poincaré Mapping, Emittance, and Energy Spread

A powerful method to visualize the orbital dynamics of phase space that might otherwise be too complex is the Poincaré mapping technique, first applied to the particle-core model by Lagniel. The idea is to sample the motion of particles at regular intervals and plot the data in a cross-section of phase space. These samples are called Poincaré surfaces of section (PSS). The study herein uses PSS extensively to illustrate halo formation and chaoticity.

Emittance is a conventional, proportionate measure of the phase-space volume subtended by the coarse-grained beam. The elliptical shape is due to the predominant

---

20 Honerkamp, op. cit., 188.
21 Bohn & Sideris, op. cit., 104202.
24 Wangler, op. cit., 259.
linear focusing forces in a linac. Root-mean-square (rms) emittance, defined from the
d second moments is
\[ \tilde{\varepsilon}_{x}^{2} = \langle (x - \langle x \rangle)^2 \rangle \left( \langle (v_x - \langle v_x \rangle)^2 \rangle - \langle x - \langle x \rangle \rangle \langle v_x - \langle v_x \rangle \rangle \right)^2, \]
which may then be expressed as
\[ \tilde{\varepsilon}_{x} = \sqrt{\langle (x^2) - \langle x \rangle^2 \rangle \left( \langle v_x^2 \rangle - \langle v_x \rangle^2 \right)^2 - \langle x v_x \rangle - \langle x \rangle \langle v_x \rangle \rangle^2}, \]
where
\[ \langle \cdots \rangle = \frac{1}{N} \sum_{i=1}^{N} (\cdots)_i. \]
The rms emittance definition is convenient because it can also be used to define an
arbitrary distribution, as it only uses rms descriptions of the beam.

Not all particles will acquire or lose the same amount of energy within a beam
bunch, therefore creating energy spread.25 This study focuses on the energy spread of
localized clumps of particles placed in specific regions of phase space and can be
expressed as
\[ \Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}. \]

25 See, e.g., Chao, A.W., *Physics of Beam Instabilities in High Energy Accelerators*
2.5 The Particle-Core Model

Several types of distributions have been used in particle-core models. Previous studies\(^\text{26}\) have considered a warm-fluid Kapchinkij-Vladimirskij (KV) distribution for its analytic simplicity, since, for instance, its real space projection has uniform density, therefore linear space-charge forces. This work has led to both theoretical and experimental investigations in beam characterization and dynamical phenomena, such as emittance growth and mechanisms of halo formation. However, the KV picture does not include effects of a full nonlinear nature, for example, the influence of a nonlinear space-charge force. Other distributions, such as parabolic, conical, and Gaussian, have also been used to represent particle distributions in beam studies.\(^\text{27}\) Hereafter, a particle-core model with a waterbag distribution\(^\text{28}\) will be considered.

The waterbag model is a reasonable alternative to the KV model since its nonlinear core force may better describe the behavior of intense beams. An important feature is that this distribution is time dependent, or made to breathe; it is no longer fixed, unlike a stationary distribution.\(^\text{29}\) A previous study considered a cylindrical breathing waterbag to understand mechanisms of halo formation in beams.\(^\text{30}\) Now, a spherical version is considered and is diagnosed in far more detail.

---


\(^{28}\) Ibid., 352.

\(^{29}\) Ibid., 347.

Herein, a spherically symmetric, homologously breathing waterbag is used to investigate the influence of time dependence in space-charge potentials. This is a two-parameter family of particle-core models: one parameter is the envelope mismatch $M$, a ratio of the initial beam radius to the radius of the ideally matched beam; the other is the tune $\eta$, a measure of effective focusing strength in the presence of space charge, or more specifically, a ratio of phase advance per focusing period with and without the presence of space charge. The length scale is the radius of the equivalent uniform matched beam, $R_0$. The beam bunch traverses a smoothed focusing channel, where only the average forces are considered. The time scale is the inverse of the angular focusing frequency, $\omega_F^{-1}$. The self-consistent\textsuperscript{31} equilibrium distribution function in six-dimensional phase space\textsuperscript{32} is

\[
f(r, p) = N[H_o-H(r, p)]^{-1/2} \text{ for } H_o>H,
\]

where $H$ denotes the Hamiltonian

\[
H(r, p) = \frac{p^2}{2m} + \frac{1}{2} kr^2 + e\Phi(r),
\]

$f(r, p)$ is zero otherwise, and $H_o$ is a constant. Invoking spherical symmetry, $f$ in terms of $r$ and the radial velocity $v$ is, for $H_o>H$:

\[
f(r, v) = \frac{N}{\sqrt{G(r)-\frac{1}{2}mv^2}}; 0 \leq v < \sqrt{\frac{2G(r)}{m}},
\]

\textsuperscript{31} “Self-consistency” refers to Poisson’s equation, where $\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}$, and $\rho = \rho\{f\}$.

\textsuperscript{32} Gluckstern & Fedotov, op. cit., 054201.
where \( G(r) \equiv H_o - \frac{1}{2}kr^2 - e\Phi(r) \). The uniform charge density \( \rho \) (see Fig. 2.2) and the potential \( \Phi \) are found by solving Poisson’s equation:\(^{33}\)

\[
\rho(r,t) = \frac{3Q}{4\pi R^3} \nu \left\{ 1 - \frac{i_0(\xi)}{i_0(\xi_0)} \right\},
\]

\[
\Phi = \begin{cases} \frac{\nu}{\xi R\alpha} - \frac{1}{2} \left( 1 - \frac{1}{R^3} \right) \left( (\xi R\alpha)^2 - x^2 \right) - \frac{\mu}{R} d_0(\frac{\xi}{R\alpha}), & |x| < \xi R\alpha; \\ -\frac{1}{2} \left( (\xi R\alpha)^2 - x^2 \right) + \frac{\nu}{|x|}, & |x| \geq \xi R\alpha, \end{cases}
\]

where \( R \equiv R(t) \) is the radius of the equivalent uniform rms-mismatched beam, and \( x \) is the position of the test particle. The time dependence is found in the envelope equation:\(^{34}\)

\[
\ddot{R} + R - \frac{\eta^2}{R^3} = 1 - \frac{\eta^2}{R^2} = 0; \quad R(0) = M, \quad \dot{R}(0) = 0.
\]

\( M \) is the rms mismatch parameter, where \( R(0) / R_0 = R(0) = M \), and \( R_0 = 1 \) is the radius of the equivalent uniform matched beam. Therefore, an ideally rms matched beam would have a mismatch of \( M = 1 \). The auxiliary quantities:\(^{35}\) are

\[
i_0(y) \equiv y^{-1} \sinh y,
\]

\[
d_0(y) \equiv i_0(\xi) - i_0(y),
\]

\[
\alpha \equiv \left\{ \frac{(3 + \xi^2)i_0(\xi) - 3 \cosh \xi}{\left[ \xi^4 + 15(2 + \xi^2) i_0(\xi) - 5(6 + \xi^2) \cosh \xi \right]} \right\}^{1/2}
\]

\[
\mu \equiv 3\alpha^2 / i_0(\xi),
\]

\[
\nu \equiv \frac{\xi\alpha^3}{i_0(\xi)} [(3 + \xi^2)i_0(\xi) - 3 \cosh \xi].
\]


\(^{34}\) Ibid.

\(^{35}\) Ibid.
Figure 2.2. Waterbag density profiles. Three initial waterbag density profiles for $M = 0.5$ (red), 1.0 (black), and 1.5 (blue).
The intermediate parameter $\xi$ is found from the tune $\eta$ by solving

$$
\eta^2 = \frac{15}{4i_0(\xi)} \left( (12 + 5\xi^2)i_0^2(\xi) - 3 \cosh \xi [3i_0(\xi) + \cosh \xi] \right) \left[ 30 + \xi^2(15 + \xi^2) \right] i_0(\xi) - 5(6 + \xi^2) \cosh \xi.
$$

The equations of motion of the test particles are found from

$$
\ddot{x} = -\partial_x \Phi,
$$

which yields\(^{36}\)

$$
\ddot{x} + x = \begin{cases} \frac{x}{R^3} - \mu(x) \frac{\cosh(x/R\alpha) - i_0(x/R\alpha)}{Rx}, & |x| < \xi R\alpha; \\ \nu(x) \frac{|x|}{x^3}, & |x| \geq \xi R\alpha. \end{cases}
$$

This model admits a significant number of both regular and chaotic orbits. In addition, we will see that the test particle trajectories, though initially placed inside the beam envelope, may readily explore regions well outside the envelope.

The spherical waterbag model has a number of advantages: first, the distribution, the charge density, and corresponding space-charge potential are all analytic. Second, the beam-size mismatch $M$ and the space-charge-depressed tune $\eta$ parameterize the family of models, with $\eta$ ranging from zero space charge, or an emittance-dominated regime ($\eta = 1$), to the space charge limit ($\eta = 0$). Third, the equilibrium configurations, which include the Hamiltonian, distribution function, charge density, and Poisson’s equation, are dynamically stable.\(^{37}\) Last, chaotic orbits arise once the stationary waterbag is made to oscillate, or breathe. Therefore, the waterbag particle-core model can be a valuable tool to investigate how time dependence affects phenomena related to chaos, as well as the production of halo.

\(^{36}\)Bohn, C.L., personal communication.

CHAPTER III
NUMERICAL METHODS

3.1 Summary

The work presented in this study is the result of an extensive analysis of several waterbag particle-core models using numerical methods. The simulation code utilized herein was solely written by I.V. Sideris and was revised as needed throughout the investigation.

A fifth-order Runge-Kutta algorithm with variable time step was used to integrate each of the particle orbits, providing the efficiency and accuracy needed for this model. A method for the calculation of the maximal short-time Lyapunov exponent is also given, which provides a credible test for chaos. An algorithm that adds colored noise to the system is also presented. Several variables were tabulated for each time step, and information regarding the statistics followed the successful completion of each simulation. With these data, a number of useful measures were calculated and plotted in order to characterize and visualize any apparent instabilities and effects of time dependence and mixing in a homologously breathing spherical waterbag model.
3.2 Integration Methods

3.2.1 The Runge-Kutta Method

The Runge-Kutta method is an extension of the Euler method for numerically integrating ordinary differential equations (ODE). This type of ODE integration provides significant accuracy without imposing a high computational burden or cost. The fifth-order Runge-Kutta algorithm with variable time step\(^{38}\) was used to integrate the orbit in phase space of test particles that populated and interacted with the particle core. Using higher order algorithms may have smaller truncation error but requires more computation, especially beyond fourth order, so that this particular method becomes a decent trade-off between efficiency and accuracy. Using a variable time step provides additional improvement over a fixed time step routine in that it works out how small the time interval should be for the integration error to be more accurate than a particular threshold. When there are complexities in the function to be integrated, this ensures that errors will be minimized, thus offering reliability and stability.

3.2.2 Lyapunov Exponents

There are several methods for calculating the maximal short-time Lyapunov exponent $\chi$, or Lyapunov characteristic number.\(^{39}\) Note that the largest of the exponents is commonly found because it helps determine the chaoticity of the entire dynamical orbit. A well-established algorithm in the field of chaotic dynamics is used here.\(^{40}\) As an averaged measure of separation between two orbits, the method of finding $\chi$ is applied as follows. An initial condition in phase space is chosen in the area of interest. Given an appropriate choice of separation, $\Delta x_0$, a second initial condition is chosen. Then, the two orbits evolve over some interval $\Delta t$ (see Figure 3.1) and the new separation $\Delta x$ is recorded. The orbits may follow an exponential trend from which $\chi$ is found:

$$\Delta x(X_0, t) = \Delta x_0 e^{\chi t} \rightarrow \chi = \lim_{t \to \infty} \lim_{\Delta x_0 \to 0} \frac{1}{t} \ln \left( \frac{|\Delta x(X_0, t)|}{|\Delta x_0|} \right).$$

Now, the distance between the two orbits must be properly rescaled in order to carry out the next iteration of evolution; this step is crucial and often error-prone. The method of choice incorporates renormalizing the distance between the two deviating orbits by bringing them close to together again (along the line which connects them) to the initial separation $\Delta x_0$ and then repeating the process of integration until the average exponent converges to an acceptably stable value.

---


Figure 3.1. Renormalization of Lyapunov exponent. The drawing illustrates the resulting deviation between the paths of two evolved orbits. Note that it is essential that the two orbits be renormalized along the trajectory of $\Delta x$ before reintegrating.

3.2.3 Generating Colored Noise

The colored noise used in this study is produced as the output of a stochastic differential equation, of which the input is a choice of a strength of Gaussian white noise as well as an autocorrelation time $t_c$. This particular technique has been used extensively in previous investigations.4¹

3.3 Simulation Parameters

The simulation integrated the two-parameter family of spherical waterbag particle-core models over 2,048 units in physical time, $\sim 583 \ t_D$, i.e., 583 space-charge-
depressed orbital periods. For each time step, the Runge-Kutta algorithm evaluated the
equations of motion while another algorithm computed the Lyapunov exponent. The
command line of the program required 12 inputs, including other parameters that were
later added as other scenarios were introduced. The inputs and parameters are:

NTEST – number of test particles (usually $10^3$ or $10^4$)
DT – time step of integration
OUTSTEP – number of steps integrated before recording in OUT
TSTOP – stop time of integration (2048 was typically used)
TRENORM – renormalization time (used for data in LOUT)
IN – name of test body’s input file (initial conditions of positions, velocities)
OUT – name of output file
POUT – name of Poincaré output file
PINFO – name of Poincaré information file
LOUT – name of the Lyapunov output file
ETA -- frequency
ALPHA – noise parameter
FRICTION – friction parameter of white noise
EPSILON* -- a parameter used to add a nonlinear focusing term
MM* -- a parameter used for an internal collective mode
OMEGA* -- a parameter used for an internal collective mode
(* These parameters were added for additional investigations.)
Typical executable command line script appeared as follows:

```
./smooth_waterbaglfinc 1000 0.01 32 2048 15 input_0.15_0.50e.dat
output_0.15_0.50_0.0000e.dat poutput_0.15_0.50_0.0000e.dat pinfo_0.15_0.50_0.0000e.dat
loutput_0.15_0.50_0.0000e.dat 0.15 0.0125 0.0000 &
```

The output files were tab delimited and grouped by particle orbit. The output file OUT contained the time step, instantaneous position, velocity, energy, and envelope calculations. The Poincaré output file POUT included the same information as OUT but recorded data at each time when the breathing mode reached a minimum amplitude, \( R_{\text{min}} = 0.50 \). This method of tabulation enabled the user to plot the resulting data using the Poincaré mapping technique. The LOUT file contained the maximal short-time Lyapunov exponent for each particle orbit. PINFO contains the total number of time steps needed for the integration of each particle orbit. ETA is the space-charge-depressed tune. The input parameters ALPHA (\( \alpha \)) and FRICTION were used to calculate the strength \( \langle |\delta \omega| \rangle \) and the autocorrelation time associated with the colored noise in this model, where \( \alpha = 1/t_c \). A wide range of noise strengths is considered, and those applied are \( \langle |\delta \omega| \rangle = 0, 2 \cdot 10^{-3}, 0.01, \) and 0.05. All of the output files also included variables calculated in auxiliary expressions during the simulation as well as input parameters for file identification. The last three parameters listed were used as different scenarios were considered. Epsilon (\( \epsilon \)) was used to quantify a cubic nonlinear focusing term to the equations of motion. MM and OMEGA (\( \Omega \)) were used to apply an internal collective mode in addition to the breathing mode due to mismatch.
CHAPTER IV
RESULTS AND ANALYSIS

4.1 Overview

The investigation began by evolving $10^4$ orbits, initially distributed according to the waterbag density profile, with various noise strengths. A mixing experiment was then used to characterize the evolution of $10^3$ orbits, initially localized “clumps” in phase space, in regular, sticky chaotic, and wildly chaotic regions. Long-lived orbits, evolved well beyond the span of a real linac but typical in synchrotron and storage rings, were utilized to exemplify transient dynamics. A nonlinear focusing force as well as an internal collective mode was also added later to explore dynamic sensitivities.

The study focused upon a space-charge-dominated beam clump, so tunes of $\eta = 0.15$ and 0.25 were used throughout the analysis. Six mismatch values, $M = 0.50, 0.75, 1.00, 1.10, 1.30,$ and $1.50$, were used to parameterize the analysis. Note that the initial radii of the density profiles varied with mismatch parameter $M$ (see Figure 2.1.) For most cases, the orbits were set into motion for $583\ t_D$. For the clumping analyses, $\eta = 0.15$ and $M = 0.50$ with a wide range of noise strengths, specifically $2 \times 10^{-4} \leq \langle |\delta \omega| \rangle \leq 0.05$, were used to carefully observe orbital behavior. The autocorrelation time used was $t_c = 80t$, or $\approx 23\ t_D$, which corresponds to appreciable changes during the evolution
of a beam bunch and can also represent the orbital period of a particle traversing a synchrotron or storage ring.

4.2 Halo Formation

Previous studies have shown that a key mechanism for halo formation is a 2:1 parametric resonance with the beam core due to a lowest order breathing mode. Does this model reveal the same result? The answer is yes!

Without a mismatch, there were neither observable chaotic orbits nor any mechanisms that would instigate halo formation. An initial mismatch, though, allowed test particles to oscillate at other frequencies and therefore resonate considerably with the core. Figure 4.1 illustrates an approximate 2:1 resonance with the envelope, although the halo is short lived. The particle orbit increases in amplitude until it is completely out of phase, then falls back to core-like amplitudes. This is because the frequency of the particle depends on the amplitude, and so the change in amplitude pushes the particle off resonance. The same figure also shows that the orbit appears to go in and out of 2:1 parametric resonance randomly, even without noise. As a comparison, regular orbits resemble beating, where the particle orbit is softly driven by higher core frequencies.

Without colored noise, there were clear, hard upper bounds to the halo amplitude for the chosen parameters. Notably, the maximum core amplitude was easily doubled by the maximum halo amplitude.
Now, what ensues from the presence of colored noise? The PSS of a single orbit with noise strength $\langle |\delta \omega| \rangle = 0.01$ that is shown in Figure 4.2 did something peculiar but expected. The orbit, resonating with the core, broke through the outer tori to amplitudes
far beyond the seemingly impenetrable hard upper bound of the halo. This observation is critical and agrees with a previous study\textsuperscript{42} that suggests the presence of colored noise can kick statistically rare particles back into phase with the core oscillation so that they move with ever-larger amplitudes.

To examine further how different regions are affected by time dependence as well as colored noise, a clumping strategy was used. Three clumps of 1,000 orbits were initially placed in three distinct regions of tightly localized phase space, either regular, “sticky” chaotic, or wildly chaotic. These regions were identified by plotting the largest Lyapunov exponent against the initial position of each orbit as seen in Figure 4.3. The

\textsuperscript{42} Bohn & Sideris, op. cit., 264801.
clumps were then set into motion for 583 $t_D$. By using PSS, Figure 4.4 shows an intriguing look at how orbits initially placed in different regions respond to resonances due to an initial beam-size mismatch. The regular orbits mix as a power law in time and their filaments eventually complete stable ellipses. The “sticky” chaotic orbits seemed initially unperturbed, but eventually “unstuck” themselves from the stable islands to explore the chaotic sea surrounding the core. Wildly chaotic orbits almost immediately migrated to fill the surrounding chaotic sea. The migration of chaotic orbits was then observed to form a halo. This type of plot illustrated how a time-dependent envelope gave rise to chaotic mixing and halo formation.

Figure 4.3. Lyapunov exponents vs. initial condition $r$, with tune $\eta = 0.15$ and mismatch $M = 0.50$, 10,000 orbits. The localized distributions of the three clumps used to illustrate contrast in evolutionary dynamics of the regular (green), “sticky” chaotic (red), and wildly chaotic (blue) were chosen at $r = 0.425, 0.475,$ and $0.525$, respectively.
Figure 4.4. Stroboscopic PSS snapshots of mixing in a breathing spherical waterbag potential with considerable space charge, $\eta = 0.15$. The first five plots illustrate evolution of three clumps, each with 1,000 test particles that were initially in tightly localized phase space (the initial velocity of each particle is zero): (blue) wildly chaotic orbits, (green) regular orbits, (red) “sticky” chaotic orbits. Note that in the fifth plot ($t = 420 \, t_D$, i.e. 420 orbital periods) there are some red particles in the “ears,” i.e., the halo; these red orbits “unstuck” themselves from their regular islands and migrated to fill the chaotic sea. For perspective, a 1 GeV proton linear accelerator would span $\sim 100 \, t_D$; i.e., this figure is meant to exemplify transient dynamics, not necessarily to represent a real machine. As a comparison, the sixth plot shows the complete PSS computed with 500 test particles that were initially distributed according to the waterbag density profile.
4.3 Halo Population

Now, how does time dependence affect halo population density? Chosen boundaries were used to separate orbital trajectories that occupied the chaotic sea around the core to count those that occupied the halo. Using the PSS of $\eta = 0.15$ and $M = 0.50$ as a guide (see Figures 4.4 or 4.9 below), any particle orbit beyond $x = 1.0$ or $v_x = 1.75$ was counted as a halo occupant for each time step as seen in Figures 4.5 and 4.6. For either plot, the growth in population appears to be linear, even for modest noise strength.

Figure 4.5 illustrates the time dependence of halo population for the clumps of the previous section. For the clump initialized in a wildly chaotic region, the halo forms within 25 to 40 $t_D$ for a wide range of noise strengths. In fact, at 100 $t_D$, the reference proton linac length, the halo population is roughly 5% regardless of a wide range in noise strengths. The only difference is observed when clumps were allowed to evolve the set length of 583 $t_D$. The clump placed in a sticky chaotic region was more sensitive to the presence of noise. With increasing noise strength, sticky chaotic orbits started entering the halo earlier, hinting at how noise diffuses the confining tori. However, only with noise strength of $\langle |\delta\omega| \rangle = 0.01$ did particles from the sticky chaotic region enter the halo before 100 $t_D$. In addition, at 583 $t_D$, the range of noise increased the number of sticky chaotic orbits in the halo from 5% to 17%.
Figure 4.5. Percentage orbits in halo vs. $t_D$. Sticky chaotic orbits (red, bottom three) and wildly chaotic orbits (blue, top three) with various noise strengths $\langle |\delta\omega| \rangle = 0, 0.002, 0.01$ corresponding to solid, dotted, and dashed lines, respectively.
Figure 4.6. Percentage orbits in halo vs. $t_D$ for two comparative waterbags. Ten-thousand orbits distributed according to the waterbag density profile for $\eta = 0.15$ and various noise strengths $\langle \delta \omega \rangle = 0, 0.002, 0.01$ corresponding to solid (blue), dotted (red), and dashed (light blue) lines, respectively. Note that the evolution of a clump of particle orbits with a tune $\eta = 0.25$ (not shown) and noise strength $\langle \delta \omega \rangle = 0.01$ gave 0.05% orbits in halo after 420 $t_D$. 
For a more realistic perspective, halo population was then used to compare the two PSS profiles that differed only by tune depression using the waterbag density profile as seen in Figure 4.6. With $\eta = 0.15$ and $M = 0.50$, halo started within 40 to 50 $t_D$ for a wide range of noise strengths. Specifically, at a reference time scale of 100 $t_D$, noise strengths of $\langle \delta \omega \rangle = 0, 0.002, 0.01$ resulted in $\sim 0.25\%$, $0.25\%$, and $0.6\%$ orbits in halo, respectively. In contrast, with $\eta = 0.25$ and only with modest noise strength $\langle \delta \omega \rangle = 0.01$ did orbits jump into the halo with no apparent linearity at $420$ $t_D$ averaging $0.05\%$ thereafter.

These results illustrate that the percentage of orbits within the halo steadily increase linearly in time with a strong tune depression that generates chaoticity. Importantly, the waterbag profile produced halo well within the span of a linac and posed much higher percentages for long-lived orbits, so designers must choose acceptable apertures and collimators to avoid this beam loss. Orbits enhanced with the presence of noise were given easier and quicker access to unconfined regions of chaos, creating even more unwanted halo. For instance, even without noise, a scenario with $\eta = 0.15$ and $M = 0.50$ appears to allow an incredible percentage ($0.25\%$) of orbits to easily exceed $R \approx 3.50$, which is seven times the initial core amplitude and double the maximum. Also, note that internal modes have a much bigger effect on the percentage of orbits in halo as well as pushing them to considerable radii, which will be shown in §4.7.1.
4.4 Mixing Experiments

The emittance was calculated at each time step for three clumps of 1,000 particle orbits initially placed in regular, sticky chaotic, and wildly chaotic regions as used in §4.2. To supplement this analysis, stroboscopic Poincaré plots exhibiting early exponential-like growth were also generated.

All three clumps evolved distinctly; their emittances are plotted in Figure 4.7. The clump placed in a regular region appears to be well fit by a power law, and the emittance stabilizes once an elliptical filament is completed as seen in the Poincaré plots in Figure 4.8. Plotting the energy spread helped illustrate the differences in fluctuations and growth within particular regions as they fill their available phase space. Both the emittance and energy spread plots in Figure 4.7, though, illustrate that both chaotic regions decompose into three distinct regions. Inspection of the sticky orbits (red) in the emittance plot reveal an abrupt change at two key events: At ~ 50 $t_D$, the orbits have filled the regular islands and begin migrating away into the chaotic sea; at ~ 220 $t_D$, sticky orbits begin to form a halo along with their wildly chaotic counterparts. Initially, the stroboscopic plots indicate that the initial growths of the sticky orbits follow a power law since the orbits are still “stuck” around the regular islands. When this clump enters the chaotic region, both the emittance and PSS plots agree that exponential growth takes place. The wildly chaotic orbits, within three $t_D$, grow exponentially much quicker but in a similar fashion.
Figure 4.7. Natural logarithm of the rms emittance \( \ln(\varepsilon_{\text{rms}}) \) (top) and energy spread \( \ln(|\Delta E|) \) (bottom) vs. \( t_D \in [0, 583] \) for the three localized clumps of 1,000 orbits each. In particular, inspection of the sticky orbits (red) in the emittance plot reveal an abrupt change at two key events (red arrows): At \( t_D \cong 50 \), the orbits have filled the regular islands and begin migrating away into the chaotic sea; at approximately 220 \( t_D \), sticky orbits begin to form a halo along with their wildly chaotic counterparts. The smaller panels help illustrate initial exponential or power law growth.
Figure 4.8. Stroboscopic plots exhibiting early exponential growth. \((v_x, vs. x; x, v_x \in [-2, 2]; t_D \in [3, 36])\) Note how the regular (green) orbits and sticky chaotic (red) orbits initially follow power law-like growth as they form a filament, completing an ellipse. In contrast, the wildly chaotic (blue) orbits quickly leave the stable islands to explore a chaotic sea, exhibiting exponential growth.
4.5 Overall Results from Colored Noise

A number of measures were utilized in order to characterize the effects of colored noise. The Poincaré mapping technique was used extensively to identify the effects due to parametric resonance and colored noise. In addition, chaoticity was compared to the initial position and mismatch parameter and their corresponding Lyapunov exponents.

To help identify the effects of time dependence, complete PSS were plotted for two comparative models of 10,000 particles with $\eta = 0.15, 0.25, M = 0.50$, and various noise strengths over $583 \, t_D$. In particular, the plots in Figure 4.9 illustrate how colored noise enhances chaotic behavior, including halo. Note how noise weakens both inner and outer tori, allowing chaotic orbits to more easily explore additional regions, engulfing various islands of regularity.

Using Lyapunov exponents, the sensitivity of the initial condition $x$ in Figure 4.10 shows that, in almost any case, the outer 15 to 20% of the density profile gave rise to chaotic orbits, a consequence of being initially placed in chaotic regions near the edge of the envelope. Colored noise, however, smears many of the orbits into chaos, even with small noise strengths, $\langle |\delta x| \rangle = 0.002$. It is clear in the plots of Figure 4.11 that the presence of colored noise has a greater impact on orbits whose mismatches deviate farthest from unity. It is also clear that a stronger space-charge-depressed tune increases the likelihood of orbits becoming chaotic.
Figure 4.9. Poincaré sections for two comparative spherical waterbags. (Top) The first three panels show the complete PSS computed for 10,000 test particles using the waterbag density profile with a strong tune depression $\eta = 0.15$. The parameters of the model were allowed to fluctuate by including a random term in the form of colored noise, $\langle |\delta \omega| \rangle = 0, 0.002, \text{and} 0.01$, respectively. This particular model admits a large number of chaotic orbits, and the presence of noise weakens the outer tori and thickens the predominant halo. (Bottom) As a comparison, the last three panels also show the complete PSS computed for the same number of test particles but with the tune depression lessened to 0.25.
Figure 4.10. Lyapunov exponents of 10,000 orbits vs. initial condition $r$, with tune $\eta = 0.15$ and mismatch $M = 0.50, 0.75$, for various noise strengths. Note how noise excited particle orbits near the core center into chaos.
Figure 4.11. Average Lyapunov exponents and percentage of chaotic orbits for samples of ten-thousand initial conditions using tunes $\eta = 0.15, 0.25$ for various mismatch parameters and various noise strengths. The noise strengths $\langle \delta \omega \rangle = 0, 0.002, 0.01, 0.05$ are designated by solid (blue), dotted (red), dashed (light blue), and dash-dotted (green.)
4.6 Transient Chaos

Single orbits were allowed to evolve at least eight times longer than the previous integrations. This is nowhere near the usual span of a linac of ~100 $t_0$, but the following long run-time integrations are meant to exemplify transient dynamics.

Figure 4.12 illustrates the instability of a chaotic orbit and the transience that evolved through time. By visual inspection, there are at least five distinguishable behavioral patterns embedded in this signal. These segments were chosen from a long-lived orbit and separated by color in the corresponding Poincaré plot. Most of the chosen orbital segments were in a steady state, even when orbits entered the halo. The orbital periods in black explored a chaotic sea and apparently entered this region between transitions to and from the islands. Figure 4.13 shows the complete PSS, illustrating that the most of the orbit (black) was chaotic.

Adding colored noise to the system not only breaks the tori but also strengthens the chaotic orbits, expanding them into new regions that were once apparently inaccessible. For example, Figures 4.9 and 4.14 show that an added noise strength of $\langle |\delta \omega| \rangle = 0.01$ clearly destroyed any binding tori, creating new configurations dominated by chaos. Interestingly, Figure 4.14 also illustrates that the presence of noise can actually suppress the amplitude of a particle orbit, kicking it consistently towards the center of the core as the orbit apparently loses more and more energy. Other observed
Figure 4.12. A particle trajectory corresponding PSS for a “sticky” chaotic orbit. (top) The trajectory showing that it spent most of the time exploring unstable regions (black). (bottom) The corresponding PSS of the trajectory without the chaotic segments to accentuate the smaller islands as well as the path of the halo.
Figure 4.13. The complete PSS of the very “sticky” chaotic trajectory in Figure 4.11. The black represents the chaotic sea where the particle orbit spent most of its time.
Figure 4.14. $x$ vs. $t_D$ for three distinct orbits, before and after the presence of noise. A regular orbit (green, top), a sticky chaotic orbit (red, middle), and a wildly chaotic orbit (blue, bottom) with (right) and without (left) modest noise $\langle |\delta \omega| \rangle = 0.01$. Notice how colored noise actually suppresses the wildly chaotic orbit amplitude.
orbits actually broke the inner tori binding the halo “ears” and continued to spiral into a trajectory to \( R \approx 2.0 \) before spiraling back out and continuing to explore its now enormous chaotic region, eventually expanding the halo beyond \( R = 5.00 \), more than three times the maximum core amplitude. This agrees with the initial hypothesis that colored noise would expel particles from the core to much larger amplitudes, but a much larger sample of orbits is needed to quantify the probability of occurrence as well as how early this could occur during a realistic time scale. Colored noise has allowed the outer tori to become easily penetrable as these orbits expand their available phase space to larger amplitudes.

### 4.7 Excursions

#### 4.7.1 Internal Collective Modes

To gain additional insight of the breathing waterbag model, modifications to the equations of motion were considered. One such scenario was the introduction of a physically sensible, though not self-consistent, implementation of an internal collective mode. In previous work,\(^\text{43}\) collective mode oscillations conspire with colored noise to produce enhanced, larger amplitude beam halo. This study found the same result.

In order to mimic this collective mode in an envelope-matched beam, the idea was to insert \( R(t) \), a new envelope radius calculated from linearizing the envelope

---

\(^{43}\) Boi\n n & Sideris, op. cit., 104202.
equation,\textsuperscript{44} into the equations of motion. Note that the boundary of the beam clump has been changed from $\xi Ra \to \xi a$ so the envelope is now constant. Upon linearization, making the change of variables

\[ R \to (1 + \rho); \rho(0) = M - 1, \frac{d\rho(0)}{dt} = 0: \]

\[ \dot{\rho} + (1 + \rho) - \frac{\eta^2}{(1 + \rho)^3} = \frac{1 - \eta^2}{(1 + \rho)^3} = 0 \to \rho(t) = (M - 1)\cos(\sqrt{3 + \eta^2} t). \]

The resulting model of $R(t)$ is

\[ R(t) = 1 + (M - 1)\cos(\sqrt{3 + \eta^2} t). \]

Integrating this collective mode into the waterbag model produced a sizable amount of chaotic orbits, given only a small degree of mismatch, and no colored noise. The addition of colored noise enhanced the instabilities even further, and indeed kicked the orbits to greater amplitudes as illustrated in Figure 4.15.

The addition of a collective mode in a waterbag configuration easily broke the confining tori of the beam, resulting in a wide range of highly chaotic orbits that made it difficult to analyze further model subtleties. However, this scenario does offer insight regarding the sensitivity of the waterbag model, as it resulted in considerably larger halo amplitude.

\textsuperscript{44} Strasburg & Davidson, op. cit., 5753.
Figure 4.15. Maximum amplitude vs. $M$, various noise strengths with a collective mode.
Another consideration was the addition of nonlinear focusing term in the Hamiltonian. The equation of motion was modified with the addition of a cubic term $\varepsilon x^3$, a consequence of adding the appropriate quartic term in the total energy of the particle:

$$E = \frac{1}{2} x^2 + \Phi(a) + \begin{cases} 
-\frac{1}{2}(1-\frac{1}{R})(a^2-x^2) - \frac{1}{4} \varepsilon (a^4-x^4) - \frac{\mu}{R} [i_0(\xi) - i_0(\frac{\varepsilon}{a})], |x| < a; \\
-\frac{1}{2} (a^2-x^2) - \frac{1}{4} \varepsilon (a^4-x^4) + \nu \left(\frac{1}{|x|} \frac{1}{a}\right), |x| > a,
\end{cases}$$

where $R = R(t)$, $a = \xi R\alpha$, and $\Phi(a) = h_0 + \frac{V}{a}$ in which $h_0$ can be set $= 0$ without loss of generality. PSS for $\eta = 0.25$, $M = 0.75$, and $\varepsilon = 0.05$ are plotted in Figure 4.16. The plot exhibits mostly stable regions; the tori are still robust despite the additional nonlinear focusing force. Interestingly, a modest amount of noise once again penetrated the outer tori and formed a halo, a similar scenario shown in Figure 4.9. A mismatch of $M = 0.50$ easily created halo without the presence of noise. Note that with $\varepsilon = -0.01$, for $\eta = 0.15, 0.25$, various mismatch, and noise strengths up to $\langle |\delta\varphi| \rangle = 0.01$ produced no halo.
Figure 4.16. PSS for $\eta = 0.25$, $M = 0.75$, $\epsilon = 0.05$, with increasing noise strengths $\langle \delta \omega \rangle = 0, 0.002, 0.01$, with a nonlinear focusing force.
CHAPTER V
CONCLUSIONS

This study analyzed a particle-core model to describe better the underlying mechanisms that lead to unwanted evolutionary dynamics such as halo formation, which leads to beam loss and radioactivation. It has been shown that time dependence alone can trigger chaotic dynamics. By increasing space-charge forces, the chaotic regions grew, allowing more particle orbits to become affected by mismatch oscillations. The model has duplicated previous results that illustrate a 2:1 parametric resonance between statistically rare particles and the core. Introducing colored noise into the system not only enhances chaotic mixing but also allows particle orbits to penetrate once-robust inner and outer tori, enabling orbits to reach new regions and amplitudes. Adding noise has also reproduced phenomena that indicate no apparent upper bound to particles that resonate with the particle core, as they are continually kicked back into such oscillations, and is particularly evident if nonbreathing internal modes are excited. Such parametric resonance and the additions of colored noise due to machine imperfections and statistical gradient errors create unwanted consequences in future high-current and high-intensity linacs. These problems need to be investigated further and pinpointed so that such losses may be mitigated by collimation or, ideally, eliminated.
Further Work

New particle models may still be needed to best describe how such underlying mechanisms of unwanted beam losses and halo formation may be avoided and/or eliminated. Future studies using particle models may include larger test particle distributions, longer epochs, a wider range of parameters, and additional excursions that lead to better representations of the dynamics of real beam clumps and their corresponding beam loss percentages due to halo.

Although the Lyapunov exponent is widely used and sufficient for the scope of this study, a problem arises while attempting to quickly and accurately identify and measure transient chaos. Because the orbits may move in and out of regular and chaotic regions, new tools need to be considered. Recently, a new, fast measure of chaos uses pattern recognition to identify transient chaos that would ordinarily not be detected with current methods.\textsuperscript{45} In addition, it has been found that, by augmenting a technique by Struckmeier and Riedel,\textsuperscript{46} an auxiliary function of an integral of motion of the time-dependent potential can be computed to reflect the near-real-time character of the orbital evolution, including transitions between regular and chaotic orbits.\textsuperscript{47} These tools may better decipher the nonlinear dynamics inherent to such configurations.

\textsuperscript{47} Bohn, C.L., personal communication.
BIBLIOGRAPHY


Contopoulos, Order and Chaos in Dynamical Astronomy (Springer, Berlin, 2002).


