1) (12 pts) True or False. Circle your answer.

(a) T F All continuous functions have derivatives.  
1 \times 1 \text{ is continuous but not differentiable at } x = 0

(b) T F All continuous functions have antiderivatives.

(c) T F If \( f(x) \) is differentiable for all \( x \) and \( f(1) = f(-1) \), then there is a number \( c \) such that \( |c| < 1 \) and \( f'(c) = 0 \).  
**Rolle's Theorem**

(d) T F If \( a \) is a critical point of \( f \), then \( f \) must have a maximum or a minimum at \( x = a \).

(e) T F If \( f'(x) = g'(x) \) for \( 0 < x < 1 \), then \( f(x) = g(x) \) for \( 0 < x < 1 \).

(f) T F \( \int_0^\pi \sec^2 x \, dx = \tan(\pi) - \tan(0) \).  
**sec^2 x is not continuous on \([0, \pi]\)**

2) (24 pts) Find the following limits. If the limit is infinite, write \( \infty \) or \( -\infty \).

(a) \( \lim_{x \to \pi^-} \csc x = \lim_{x \to \pi^-} \frac{1}{\sin x} = +\infty \)

(b) \( \lim_{x \to 0} \frac{\sin(2x)}{3x + 4x^2} = \lim_{x \to 0} \frac{2 \sin(2x)}{2x(3 + 4x)} = \lim_{x \to 0} \frac{2 \sin 2x}{2x} \cdot \frac{2}{3 + 4x} = 1 \cdot \frac{2}{3} = \frac{2}{3} \)

(c) \( \lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{3x - 1} = \lim_{x \to \infty} \frac{x \sqrt{1 + \frac{1}{x^2}}}{3x - 1} = \lim_{x \to \infty} \frac{x \sqrt{1 + \frac{1}{x^2}}}{3x - 1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{3 - \frac{1}{x}} = \frac{\sqrt{1}}{3} = \frac{2}{3} \)

(Done the long way)

(d) \( \lim_{x \to 1} \frac{x^2 - 1}{\sqrt{5x + 1} - \sqrt{7 - x^2}} = \lim_{x \to 1} \frac{(x-1)(x+1)}{\sqrt{5x + 1} - \sqrt{7 - x^2}} \cdot \frac{\sqrt{5x + 1} + \sqrt{7 - x^2}}{\sqrt{5x + 1} + \sqrt{7 - x^2}} = \lim_{x \to 1} \frac{(x-1)(x+1)(\sqrt{5x + 1} + \sqrt{7 - x^2})}{(5x + 1) - (7 - x^2)} = \lim_{x \to 1} \frac{(x-1)(x+1)(\sqrt{5x + 1} + \sqrt{7 - x^2})}{(5x + 1) - (7 - x^2)} = \lim_{x \to 1} \frac{(x^2 + 5x - 6)}{(x-1)(x+6)} = \frac{4\sqrt{6}}{7} \)
(3) (9 pts) Let
\[ f(x) = \begin{cases} 
  x^2 & \text{if } x < 0 \\
  3x & \text{if } 0 \leq x < 1 \\
  5 & \text{if } x = 1 \\
  2x + 1 & \text{if } x > 1 
\end{cases} \]

(a) Find \( \lim_{x \to 1} f(x) \) if it exists.

\[ \text{Check:} \]
\[ (1) \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x = 3 \quad (2) \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x + 1 = 3 \]

Yes, \( \lim_{x \to 1} f(x) = 3 \)

(b) Is \( f(x) \) continuous at \( x = 1 \)? Explain.

No, \( \lim_{x \to 1} f(x) \neq f(1) \)

3 ≠ 5

(c) Is \( f(x) \) continuous at \( x = 0 \)? Explain.

\[ (1) f(0) = 3(0) = 0 \quad (2) \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3x = 0 \quad (3) \lim_{x \to 0^+} f(x) = f(0) \]

Yes, \( f \) is cont. at \( x = 0 \)

(4) (12 pts) Let \( f(x) = \frac{x}{x + 2} \). Use the limit definition of derivative (either version) to find \( f'(3) \). [No credit for any other method.]

\[ (1) f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h}{x + h + 2} - \frac{x}{x + 2} \cdot \frac{(x+h)(x+2)}{(x+h+2)(x+2)} \]

\[ = \lim_{h \to 0} \frac{(x+h)(x+2) - x(x+h+2)}{h(x + h + 2)(x+2)} \]

\[ = \lim_{h \to 0} \frac{x^2 + 2x + h + 2h - x^2 - xh - 2x}{h(x+h+2)(x+2)} \]

\[ = \lim_{h \to 0} \frac{2h}{h(x+h+2)(x+2)} \]

\[ = \lim_{h \to 0} \frac{2}{(x+h+2)(x+2)} = \frac{2}{(3+2)^2} = \frac{2}{25} \]

(2) \( f'(3) = \frac{2}{(3+2)^2} = \frac{2}{25} \)
(5) (9 pts) In each part, the graph of a function is given. Draw a graph of the derivative of this function in the adjacent coordinate system.

(a) Graph is a parabola

(b)

(c) The lines are parallel
(6) (14 pts) Find the derivative of the following functions [You do not need to simplify]:

(a) \( f(x) = \sqrt{\sin(x^4 + 16)} = (\sin(x^4 + 16))^{1/2} \)

\[
f'(x) = \frac{1}{2} (\sin(x^4 + 16))^{-1/2} \cdot \cos(x^4 + 16) \cdot (4x^3)
\]

(b) \( f(x) = \frac{x^7 + \frac{1}{x^2}}{2x - 7} = \frac{x^7 + 9x^{-2}}{2x - 7} \)

\[
f'(x) = \frac{(2x - 7)(7x^6 - 8x^{-3}) - (x^7 + 9x^{-2})(2)}{(2x - 7)^2}
\]

(7) (8 pts) Find \( \frac{d^2y}{dx^2} \) if \( y = x \sec x \).

\[
\frac{dy}{dx} = x \sec x \tan x + 1 \sec x
\]

\[
\frac{d^2y}{dx^2} = x \sec x (\tan x)' + x (\sec x)' \tan x + (x)' \sec x \tan x
\]

\[
= x \sec x \cdot \sec^2 x + x \sec x \tan x \cdot \tan x + 1 \sec x \tan x + \sec x \tan x
\]

\[
= x \sec^3 x + x \sec x \tan^2 x + 2 \sec x \tan x
\]

or \( \frac{dy}{dx} = \sec x (x \tan x + 1) \)

\[
\frac{d^2y}{dx^2} = \sec x (x \cdot \sec^2 x + 1 \cdot \tan x) + \sec x \tan x (x \tan x + 1)
\]
(8) (10 pts) Find the tangent line to the curve \( xy^3 - 4xy = -6 \) at the point \((2,1)\).

Differentiate: \[
\frac{d}{dx} \left[ xy^3 - 4xy \right] = \frac{d}{dx} \left[ -6 \right]
\]

\[ x \cdot 3y^2 \cdot y' + 1y^3 - 4xy' - 9y = 0 \]

\[ 3xy^2 \cdot y' - 4xy' = 4y-y^3 \]

\[ y' \left( 3xy^2 - 4x \right) = 4y-y^3 \]

\[ y' = \frac{4y-y^3}{3xy^2 - 4x} \]

Plug in \(x=2, y=1\)

\[ y' = \frac{4(1)-1^3}{3(2)(1)^2 - 4(1)} = \frac{3}{-2} = -\frac{3}{2} \]

Tangent Line: \[ y-1 = -\frac{3}{2}(x-2) \rightarrow y = -\frac{3}{2}x + 4 \]

(9) (10 pts) Find the critical points of the function \( f(x) = x + 2 \sin x \) on the interval \([0, 2\pi]\).

Determine whether each critical point is a local maximum, minimum, or neither.

\[ f'(x) = 1 + 2 \cos x \rightarrow 1 + 2 \cos x = 0 \]

\[ \cos x = -\frac{1}{2} \]

\[ x = \frac{2\pi}{3}, \frac{4\pi}{3} \]

\[ \begin{array}{ccc}
\frac{0}{3} & 1 & \frac{2\pi}{3} \\
\frac{2\pi}{3} & + & - \\
\frac{4\pi}{3} & + & 2\pi
\end{array} \]

Local Max: \( (\frac{2\pi}{3}, \approx 3.827) \)

Local Min: \( (\frac{4\pi}{3}, \approx 2.457) \)
(10) (12 pts) The graph of the function \( y = f(x) \) satisfies all of the following conditions:

- vertical asymptote \( x = -1 \)
- horizontal asymptote \( y = 1 \)
- \( f(0) = 0 \)
- \( f'(x) = \frac{2x}{(x + 1)^3} \)
- \( f''(x) = \frac{2 - 4x}{(x + 1)^4} \)

(a) Where is \( f(x) \) increasing?

\[
\begin{align*}
2x &= 0, \quad (x+1)^3 = 0 \\
x &= 0, \quad x = -1
\end{align*}
\]

(b) Where is \( f(x) \) concave up?

\[
\begin{align*}
2 - 4x &= 0, \quad (x+1)^4 = 0 \\
x &= \frac{1}{2}, \quad x = -1
\end{align*}
\]

(c) Sketch the graph of \( y = f(x) \). Display clearly the concavity of the curve, where it is rising and where it is falling. Label all asymptotes, local extrema, and inflection points.
(11) (12 pts) A kite 100 feet above the ground is flying horizontally at a speed of 8 ft/sec. At what rate is the angle \( \theta \) between the string and the horizontal decreasing when 200 feet of string have been let out?

![Diagram of kite and string]

Formula: \( \tan \theta = \frac{100}{x} \)

Differentiate: \( \sec^2 \theta \cdot \frac{d\theta}{dt} = -\frac{100}{x^2} \cdot \frac{dx}{dt} \)

Find \( x \):

\[
x^2 + 100^2 = 200^2
\]

\[
x^2 = 30000
\]

\[
x = \sqrt{30000} \approx 173.21
\]

*Note: \( \sec \theta = \frac{200}{x} = \frac{200}{173.21} \)

\[
\left(\frac{200}{173.21}\right)^2 \cdot \frac{d\theta}{dt} = \frac{-100}{173.21} \cdot 8
\]

\[
\frac{d\theta}{dt} = -0.02 \text{ rad/sec}
\]

(12) (6 pts) Let \( f(x) = x^5 - x - 1 \). Use Newton’s method with the initial approximation \( x_1 = 0 \) to find \( x_2 \) and \( x_3 \).

\[
f'(x) = 5x^4 - 1
\]

Method \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \)

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = -\frac{-1}{-1} = -1
\]

\[
x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{-1/4}{-4} = -\frac{3}{4}
\]
(13) (12 pts) Find two numbers \(x\) and \(y\) whose difference is 100 and whose product is a minimum.

- Given: \(x - y = 100\)
- Objective: Minimize \(P = xy\)
- Rewrite \(y\) in terms of \(x\): \(y = x - 100\)
- Objective: Minimize \(P = x(x - 100) = x^2 - 100x\)
- Find \(P'\): \(P' = 2x - 100\)
- Solve \(P' = 0\)
  \[2x - 100 = 0\]
  \[2x = 100\]
  \[x = 50\]

\[\begin{array}{cccc}
- & - & + & + \\
50 & & & \\
\end{array}\]

- \(x = 50\), \(y = 50 - 100 = -50\)

(14) (6 pts) Approximate \(\int_{0}^{2} \frac{1}{1 + x^2} \, dx\) using the Riemann sum with \(n = 4\) rectangles and right-hand endpoints. Is the approximation an over-estimate or an under-estimate?

- \(f(x) = \frac{1}{1 + x^2}\), \(a = 0\), \(b = 2\), \(n = 4\)
- \(\Delta x = \frac{2 - 0}{4} = \frac{1}{2}\), \(x_0 = 0\), \(x_1 = \frac{1}{2}\), \(x_2 = 1\), \(x_3 = \frac{3}{2}\), \(x_4 = 2\)
- \(R_4 = \Delta x \left[ f(x_1) + f(x_2) + f(x_3) + f(x_4) \right] = \frac{1}{2} \left[ f(\frac{1}{2}) + f(1) + f(\frac{3}{2}) + f(2) \right] = \frac{1}{2} \left[ 0.8 + 0.5 + 0.3077 + 0.2 \right] = 0.90385\)
(15) (10 pts) For $x \geq 0$, define $g(x) = \int_0^x f(t) \, dt$, where $f$ is the function graphed below.

(a) What is $g(0)$?

$$g(0) = \int_0^0 f(t) \, dt = 0$$

(b) What is $g(1)$?

$$g(1) = \int_0^1 f(t) \, dt = 2$$

(c) What is $g'(x)$?

$$g'(x) = f(x)$$

(d) Is $g(x)$ decreasing or increasing?

Since $f(x) > 0$, $g'(x) > 0$. This means $g(x)$ is increasing.

(e) Is $g(x)$ concave up or concave down?

$g''(x) = f''(x)$. From the graph, $f''(x) < 0$ (decreasing)

so $g''(x) < 0$. $g(x)$ is concave down.

(16) (6 pts) Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $f(x, y) = \frac{xy}{x - y}$. 

(17) (28 pts) Compute the following integrals:

(a) \[ \int_{1}^{4} \frac{x^2 - \sqrt{x}}{x} \, dx = \int_{1}^{4} \frac{x^2}{x} - \frac{\sqrt{x}}{x} \, dx = \int_{1}^{4} x - x^{-1/2} \, dx \]

\[ = \left. \frac{1}{2}x^2 - 2x^{1/2} \right|_{1}^{4} \]

\[ = \left[ \frac{1}{2}(4)^2 - 2(4)^{1/2} \right] - \left[ \frac{1}{2}(1)^2 - 2(1)^{1/2} \right] = 4 + \frac{3}{2} = \frac{11}{2} \]

(b) \[ \int_{0}^{\pi/4} \frac{2 \tan \theta - 3 \sec^3 \theta}{\sec \theta} \, d\theta = \int_{0}^{\pi/4} \frac{2 \tan \theta}{\sec \theta} - \frac{3 \sec^2 \theta}{\sec \theta} \, d\theta \]

\[ = \int_{0}^{\pi/4} 2 \sin \theta - 3 \sec \theta \, d\theta \]

\[ = -2 \cos \theta - 3 \tan \theta \bigg|_{0}^{\pi/4} = \left[ -2 \cos \left( \frac{\pi}{4} \right) - 3 \tan \left( \frac{\pi}{4} \right) \right] - \left[ -2 \cos(0) - 3 \tan(0) \right] \]

\[ = -\sqrt{2} - 3 + 2 = -\sqrt{2} - 1 \]

(c) \[ \int \cos^4(3x) \sin(3x) \, dx \]

let \( u = \cos(3x) \)

\[ du = -3 \sin(3x) \cdot 3 \, dx \]

\[ \frac{1}{3} \int -3u^4 \, du = \frac{1}{15} u^5 + C \]

\[ \frac{1}{3} \int -3 \sin(3x) \, dx = -\frac{1}{15} \cos^5(3x) + C \]

(d) \[ \int t \sqrt{t^2 + 9} \, dt \]

let \( v = t^2 + 9 \)

\[ dv = 2t \, dt \]

\[ \frac{1}{2} \int \sqrt{v} \, dv = \frac{1}{2} \int v^{1/2} \, dv = \frac{1}{3} v^{3/2} + C \]

\[ = \frac{1}{3} (t^2 + 9)^{3/2} + C \]